

RICCI FLOW ON ASYMPTOTICALLY EUCLIDEAN MANIFOLDS

YU LI

ABSTRACT. In this paper, we prove that if an asymptotically Euclidean manifold (M^n, g) under the condition that $R \geq 0$ has long time existence of Ricci flow, the mass of (M^n, g) is nonnegative. In addition, we give an independent proof of positive mass theorem in dimension 3.

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1. INTRODUCTION

A smooth orientable Riemannian manifold (M^n, g) ($n \geq 3$) is called an *Asymptotically Euclidean (AE)* manifold if for some compact $K \subset M^n$, $M^n \setminus K$ consists of a finite number of components E_1, \dots, E_k such that for each E_i there exists a C^∞ diffeomorphism $\Phi_i : E_i \rightarrow \mathbb{R}^n \setminus B(0, R_i)$ such that under this identification,

$$(1.1) \quad g_{ij} = \delta_{ij} + O(r^{-\sigma_i}), \quad \partial^{|k|} g_{ij} = O(r^{-\sigma_i - k})$$

for any partial derivative of order k as $r \rightarrow \infty$, where r is the Euclidean distance function. We call the positive number σ_i the order of end E_i .

The ADM-mass [4] from general relativity of an AE manifold (M, g) is defined as

$$m(g) = \lim_{r \rightarrow \infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) dA^j,$$

where $dA^j = \partial_j \lrcorner dx$.

The general positive mass conjecture is the following, see [24, Theorem 10.1].

Date: March 16, 2016.

Conjecture 1.1 (Positive Mass Conjecture). *Let (M^n, g) be an AE manifold of dimension $n \geq 3$ with the order $\sigma > (n - 2)/2$, and nonnegative integrable scalar curvature. Then $m(g) \geq 0$ with equality if and only if $(M, g) = (\mathbb{R}^n, g_E)$.*

Henceforth, we use g_E to denote the Euclidean metric. With all those assumptions, the ADM mass is finite and independent of AE coordinates, see [5].

The positive mass conjecture was first proved by Schoen and Yau [37] in dimension three using minimal surface and its stability inequality. In addition, Schoen and Yau showed that their argument extended to dimensions less than eight by using a minimal hypersurface argument which was inductive on dimension. However, it turns out that minimal hypersurfaces may have singularities on a subset whose codimension is at least seven. Meanwhile, Witten gave a proof in [41] of the positive mass conjecture that worked in all dimensions, but only for spin manifolds. Moreover, in dimension three, Huisken and Ilmanen [19] proved the positive mass conjecture by using the technique of the inverse mean curvature. Recently, Hein and LeBrun gave a proof of the positive mass conjecture for Kähler AE manifolds, see [21]. To the author's knowledge, there is no solution for the positive mass conjecture of general dimension.

A natural question is that can we prove the positive mass conjecture by using other geometric flows, at least in dimension three? Since Ricci flow is one of the most powerful geometric flows by using which Perelman completely classified all compact 3-manifolds, see [31, 32, 33], it is interesting to know how Ricci flow interacts with AE manifolds and mass.

It was proved by Dai and Ma in [16] that Ricci flow preserves ALE condition, nonnegative integrable scalar curvature and mass. Moreover, Robert Haslhofer in [35] constructed a mass-decreasing flow which performs a conformal transformation at each surgery time.

One of the main theorem of this paper shows that if we have long time existence of Ricci flow, the AE manifold will converge to the Euclidean space in the weighted space. The fact that the limit manifold at infinity is Euclidean space is motivated by considering the steady soliton on ALE manifold, see Appendix. The convergence at infinity does not preserve the mass, but decrease it to zero.

We assume throughout this paper that the scalar curvature R is nonnegative and integrable, the manifold has only one end E ¹ and the order of the end $\sigma > (n - 2)/2$. Moreover, we fix a positive smooth function $r(x)$ on M such that $r(x) = |x|$ when $x \in E$.

Note that when $\sigma > n - 2$, $\partial_i g_{ij} - \partial_j g_{ii} = O(r^{-\sigma-1})$ and hence

$$|m(g)| \leq C \lim_{r \rightarrow \infty} r^{n-2-\sigma} = 0$$

Therefore, we can also assume that the order $\sigma \leq n - 2$.

Theorem 1.2. *Let (M^n, g) be an AE manifold with above assumptions. If there exists a solution $g(t)$ ($0 \leq t < \infty$) of the Ricci flow with $g(0) = g$, then the mass $m(g) \geq 0$ with the equality if and only if $(M^n, g) = (\mathbb{R}^n, g_E)$.*

¹In fact, all the arguments below apply to the multi-end case with slight modifications.

For the general Ricci flow, it is possible that the metric will become singular at some finite time. In the case of dimension three, we can continue Ricci flow by surgery. We prove that mass and other related conditions are preserved under Ricci flow with surgery. Moreover, if we choose surgery parameter function $\delta(t)$ small enough, there are only finitely many surgeries. We prove the finiteness of surgery times by carefully examining the change of Perelman's μ -functional over surgery times. Therefore, we have the long time existence of Ricci flow after the last surgery time and by using the result of Theorem 1.2 we can prove positive mass theorem in dimension three.

Theorem 1.3. *When $n = 3$, the mass $m(g) \geq 0$ with the equality if and only if $(M^3, g) = (\mathbb{R}^3, g_E)$.*

For the remainder of the paper, the constant C in the estimates may vary line by line.

2. MASS UNDER RICCI FLOW

We prove in this section that Ricci flow preserves AE condition and mass is preserved under Ricci flow. Different from the argument of Dai and Ma in [16], we fix an AE coordinate system along the flow. The main tool we use is the following maximum principle on noncompact manifold for Ricci flow, see [11, Theorem 12.14].

Theorem 2.1. *Suppose that $g(t)$, $t \in [0, T]$, is a complete solution to the Ricci flow on a noncompact manifold M with $|Rm(g(t))| \leq k_0$ for some $k_0 > 0$. Let*

$$Lu = u_t - \Delta u - \langle X(t), \nabla u \rangle - G(u, t),$$

where $X(t)$ is a smooth family of bounded vector fields and the function $G : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ is locally Lipschitz in the \mathbb{R} factor and continuous in the $[0, T]$ factor. Suppose that u is a smooth function such that

$$Lu \leq 0 \quad \text{and} \quad |u(x, t)| \leq \exp(b d_{g(t)}(O, x) + 1)$$

for some constant b . For any $c \in \mathbb{R}$, let $U(t)$ be the solution to the corresponding ordinary differential equation:

$$\begin{aligned} \frac{dU}{dt} &= G(U, t), \\ U(0) &= c. \end{aligned}$$

If $u(x, 0) \leq c$ for all $x \in M$, then we have

$$u(x, t) \leq U(t)$$

for all $x \in M$ and $t \in [0, T]$ as long as the ODE exists.

Theorem 2.2. *Let $(M, g(t))$, $0 \leq t \leq T$ be a Ricci flow solution with bounded curvature on M and $(M, g(0))$ is an AE manifold of order $\sigma > 0$, then*

- (1) *AE condition is preserved with the same AE coordinates and order.*
- (2) *If $\sigma > (n - 2)/2$ and R is integrable, the mass is unchanged.*

Proof. (1): Since $(M, g(0))$ is an AE manifold, there exists an end E and C^∞ diffeomorphism $\Phi : E \rightarrow \mathbb{R}^n \setminus B(0, R)$ such that under this coordinates,

$$(2.1) \quad g_{ij} = \delta_{ij} + O(r^{-\sigma}), \quad \partial^{|k|} g_{ij} = O(r^{-\sigma-k})$$

for any $k = 1, 2, \dots$.

From this condition, it is easy to conclude that $|\nabla^k \text{Rm}(0)| = O(r^{-\sigma-k-2})$.

Since the Riemannian curvature is uniformly bounded on $[0, T]$, there exists an $S > 0$ such that $|\text{Rm}| \leq S$ on $M \times [0, T]$. Now we consider the evolution equation of $|\text{Rm}|^2$ which satisfies

$$\partial_t |\text{Rm}|^2 \leq \Delta |\text{Rm}|^2 + 16 |\text{Rm}|^3 \leq \Delta |\text{Rm}|^2 + 16S |\text{Rm}|^2.$$

Let $u = |\text{Rm}|^2 e^{-16St}$, then $\partial_t u \leq \Delta u$ on $M \times [0, T]$.

Next we prove u has the same decaying condition as $u(0)$, see also [16].

Let $h(x) = r^{4+2\sigma}$ on M . We set $w = hu$ and it satisfies

$$(\partial_t - \Delta)w = Bw - 2\nabla \log h \nabla w$$

on $M \times [0, T]$ where $B = \frac{2|\nabla h|^2 - h\Delta h}{h^2}$ is uniformly bounded since $|\text{Rm}| \leq S$.

Then from above equation and maximum principle, we conclude $|w| \leq C$ and hence $|\text{Rm}| \leq Cr^{-2-\sigma}$ on $M \times [0, T]$.

Claim:

$$(2.2) \quad |\nabla^k \text{Rm}| \leq Cr^{-2-k-\sigma}.$$

Proof of the claim: We assume that the claim holds for all $0 \leq l < k$. Let $h_k = r^{4+2\sigma+2k}$ and $w_k = h_k |\nabla^k \text{Rm}|^2$, then from the evolution equation of $|\nabla^k \text{Rm}|^2$

$$(2.3) \quad \begin{aligned} \partial_t |\nabla^k \text{Rm}|^2 &= \Delta |\nabla^k \text{Rm}|^2 - 2|\nabla^{k+1} \text{Rm}|^2 + \sum_{l=0}^k \nabla^l \text{Rm} * \nabla^{k-l} \text{Rm} * \nabla^k \text{Rm} \\ &\leq \Delta |\nabla^k \text{Rm}|^2 + C \sum_{l=0}^k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| \end{aligned}$$

we have

$$(2.4) \quad (\partial_t - \Delta)w_k \leq B_k w_k - 2\nabla \log h_k \nabla w_k + C \sum_{l=0}^k h_k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}|$$

where $B_k = \frac{2|\nabla h_k|^2 - h_k \Delta h_k}{h_k^2}$ is uniformly bounded.

Moreover, by induction we have

$$h_k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| = h_k |\text{Rm}| |\nabla^k \text{Rm}|^2 \leq C w_k$$

for $l = 0$ or $l = k$ and

$$h_k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| \leq h_k r^{-4-2\sigma-k} |\nabla^k \text{Rm}| = r^k |\nabla^k \text{Rm}| \leq C w_k^{1/2}$$

for $0 < l < k$.

From (2.4) we have

$$(\partial_t - \Delta)w_k \leq -2\nabla \log h_k \nabla w_k + C(w_k + w_k^{1/2}).$$

From Theorem 2.1, we conclude that w_k is uniformly bounded on $M \times [0, T]$ since the the solution of the following ODE

$$(2.5) \quad \begin{aligned} \frac{d\phi}{dt} &= C(\phi + \phi^{1/2}), \\ \phi(0) &= c \end{aligned}$$

is bounded on $[0, T]$. Therefore $|\nabla^k \text{Rm}| \leq Cr^{-2-k-\sigma}$.

For any vector field U on M , we have

$$(2.6) \quad \begin{aligned} &|\log g(t, x)(U, U) - \log g(0, x)(U, U)| \\ &= \left| \int_0^t \frac{-2\text{Ric}(s, x)(U, U)}{g(s, x)(U, U)} ds \right| \leq C \int_0^t |\text{Rm}| ds \leq Cr^{-\sigma-2}. \end{aligned}$$

Therefore

$$(2.7) \quad g(t)(U, U) = g(0)(U, U)(1 + O(r^{-2-\sigma})),$$

and in particular,

$$(2.8) \quad \begin{aligned} g_{ii}(t) &= g_{ii}(0)(1 + O(r^{-2-\sigma})) = (1 + O(r^{-\sigma}))(1 + O(r^{-2-\sigma})) \\ &= 1 + O(r^{-\sigma}) \end{aligned}$$

Now using the identity $g_{ij} = \frac{1}{4}(g_{i+j, i+j} - g_{i-j, i-j})$ and (2.7), we conclude that $g_{ij}(t) = O(r^{-\sigma})$ when $i \neq j$.

Now from the evolution equation of the Christoffel symbol

$$\partial_t \Gamma_{ij}^k = -g^{kl}(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij})$$

and (2.2), we conclude that $\Gamma_{ij}^k = O(r^{-\sigma-3})$ and hence $\partial_i R_{jk} = O(r^{-\sigma-3})$ from the relation $\nabla_i R_{jk} = \partial_i R_{jk} - \Gamma_{ij}^l R_{lk} - \Gamma_{ik}^l R_{jl}$.

One can argue like above that $\partial_i g_{jk}(t) = O(r^{-\sigma-1})$ since $\partial_t(\partial_i g_{jk}) = -2\partial_i R_{jk}$. Now by induction we can show for all k , $\partial^{[k]} g_{ij} = O(r^{-\sigma-k})$ therefore we have shown that $(E, g_{ij}(t))$ is an AE coordinate system with the same order σ .

(2): From the definition of mass

$$m(g(t)) = \lim_{r \rightarrow \infty} \int_{S_r} (\partial_i g_{ij}(t) - \partial_j g_{ii}(t)) dA^j.$$

Since we compute the mass under the same coordinates, we have

$$\begin{aligned} m'(g(t)) &= \lim_{r \rightarrow \infty} \int_{S_r} (\partial_i g'_{ij}(t) - \partial_j g'_{ii}(t)) dA^j \\ &= \lim_{r \rightarrow \infty} -2 \int_{S_r} (\partial_i R_{ij}(t) - \partial_j R_{ii}(t)) dA^j \\ &= \lim_{r \rightarrow \infty} -2 \int_{S_r} (\nabla_i R_{ij}(t) - \nabla_j R(t)) dA^j \\ &= \lim_{r \rightarrow \infty} \int_{S_r} \nabla_j R(t) dA^j. \end{aligned}$$

Now from [26, Lemma 11], we have

$$\lim_{r \rightarrow \infty} \int_{S_r} |\nabla R(t)| dS = 0$$

for $t > 0$, so $m'(g(t)) = 0$ for $t > 0$.

On the other hand, it is easy to show $m(g(t))$ is continuous at 0, see [26, Corollary 12], hence the mass is unchanged. \square

Remark 2.3. The proof of Theorem 2.2 actually shows that if $g_{ij}(0) - \delta_{ij} \in C_{-\sigma}^k$, then $g_{ij}(t) - \delta_{ij} \in C_{-\sigma}^{k-2}$ for any integer $k \geq 4$ and $t > 0$. In addition, using the argument in [16] we can prove if $g_{ij}(0) - \delta_{ij} \in C_{-\sigma}^2$, then $g_{ij}(t) - \delta_{ij} \in C_{-\sigma}^{1,\alpha}$ for $t > 0$. The definition of the weighted space can be found in Section 5.

We know from the evolution equation of the scalar curvature $R : \partial_t R = \Delta R + 2|Rc|^2 \geq \Delta R$ and the strong maximum principle, either $R(t) \equiv 0$ for all $t \geq 0$ so that $g(0)$ is Ricci-flat and hence Euclidean by comparison geometry, or $R(t) > 0$ for all $t > 0$. Since mass is unchanged under Ricci flow, by a time translation if necessary, we can assume $R(0) > 0$ at the beginning.

3. PERELMAN'S μ -FUNCTIONAL

Recall that Perelman's \mathcal{W} entropy [31] is defined as

$$(3.1) \quad \mathcal{W}(g, f, \tau) = \int (\tau(|\nabla f|^2 + R) + f - n) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV$$

for smooth function f and $\tau > 0$. Let $u = e^{-f/2}$, (3.1) becomes

$$(3.2) \quad \overline{\mathcal{W}}(g, u, \tau) = \int (\tau(4|\nabla u|^2 + Ru^2) - u^2 \log u^2 - nu^2) (4\pi\tau)^{-n/2} dV$$

Moreover, For a general (possibly incomplete) Riemannian manifold (M, g) , μ functional is defined as

$$(3.3) \quad \mu(g, \tau) = \inf \{ \overline{\mathcal{W}}(g, u, \tau) \mid u \in W_0^{1,2}(M) \text{ and } \int_M u^2 (4\pi\tau)^{-n/2} dV = 1 \}.$$

Note that when M is complete, $W^{1,2}(M) = W_0^{1,2}(M)$. Moreover, from the definition we have $\mu_U(g, \tau) \geq \mu_M(g, \tau)$ for any open set $U \subset M$.

We have the following monotonicity result under Ricci flow for the complete non-compact manifold as compact case,

$$\mu(g(t_2), \tau(t_2)) \geq \mu(g(t_1), \tau(t_1))$$

for all $0 \leq t_1 \leq t_2 < \bar{\tau}$ where $\tau(t) = \bar{\tau} - t$, $0 < \bar{\tau} < T$ and Ricci flow exists for $[0, T]$.

The proof of the monotonicity formula is similar to the compact manifold case. We can do integration by parts by analyzing the heat kernel and the solution of conjugate heat equation, see for example [14].

It was proved in [39], that $\mu(g, \tau)$ is finite if g has bounded geometry, that is, the curvature is bounded and the injective radius is positive. In particular, for any AE manifold the μ functional is finite.

It was proved in [43] that for a manifold with bounded geometry, $\overline{W}(g, u, 1)$ has a smooth positive minimizer if $\mu(g, 1)$ is less than the corresponding value at infinity. To be precise, if for any sequence $p_n \rightarrow \infty$ on the manifold M such that (M, g, p_n) converges smoothly in the Cheeger-Gromov sense to $(M_\infty, g_\infty, p_\infty)$ and $\mu_M(g, 1) < \mu_{M_\infty}(g_\infty, 1)$, then $\mu_M(g, 1)$ has a smooth positive minimizer.

In the case of Euclidean space, we have the following type of logarithmic Sobolev inequality, which appears in the work by Weissler [17] and is equivalent to the Gross logarithmic inequality [18].

Theorem 3.1. *For any function $w \in W^{1,2}(\mathbb{R}^n)$ and $\int w^2 dx = 1$, we have*

$$(3.4) \quad \int w^2 \log w^2 dx \leq \frac{n}{2} \log \left(\frac{2}{\pi n e} \int |\nabla w|^2 dx \right).$$

Equality holds if and only if $w^2 = (4\pi\tau)^{-n/2} e^{-\frac{|x-p|^2}{4\tau}}$ for some $p \in \mathbb{R}^n$ and $\tau > 0$.

It is immediate from (3.4) that $\overline{W}(g_E, u, \tau) \geq 0$ where equality holds if and only if $u^2 = e^{-\frac{|x-p|^2}{4\tau}}$ for some $p \in \mathbb{R}^n$. Therefore, $\mu_{\mathbb{R}^n}(g_E, 1) = 0$. For an AE manifold M^n , we have $(M, g, p_n) \xrightarrow{C^\infty} (\mathbb{R}^n, g_E, p_\infty)$ for any sequence $p_n \rightarrow \infty$ by Cheeger-Gromov compactness theorem. Therefore $\overline{W}(g, u, \tau)$ has a smooth positive minimizer if $\mu(g, \tau) = \mu(\tau^{-1}g, 1) < 0$ from the above result. Note that $\tau^{-1}g$ is still an AE metric.

We have the following lemma.

Lemma 3.2. *Assume that (M_i, g_i) converges to (M_∞, g_∞) smoothly in the Cheeger-Gromov sense and $\mu(g_\infty, \tau)$ is finite, then*

$$\mu(g_\infty, \tau) \geq \limsup_{i \rightarrow \infty} \mu(g_i, \tau).$$

Proof. For any $\epsilon > 0$, we can find a $u \in W_0^{1,2}(M_\infty)$ such that $\overline{W}(g_\infty, u, \tau) \leq \mu(g_\infty, \tau) + \epsilon$. For large i , we can find $u_i \in W_0^{1,2}(M_i)$ which are the pull-back functions of u and $\lim_{i \rightarrow \infty} \overline{W}(g_i, u_i, \tau) = \overline{W}(g_\infty, u, \tau)$ by the convergence.

Therefore we have

$$\limsup_{i \rightarrow \infty} \mu(g_i, \tau) \leq \lim_{i \rightarrow \infty} \overline{W}(g_i, u_i, \tau) \leq \mu(g_\infty, \tau) + \epsilon.$$

Since the above holds for any $\epsilon > 0$, we have $\limsup_{i \rightarrow \infty} \mu(g_i, \tau) \leq \mu(g_\infty, \tau)$. □

It follows immediately from the above lemma that $\mu(g, \tau) \leq 0$ for any AE manifold since $(M, g, p_n) \xrightarrow{C^\infty} (\mathbb{R}^n, g_E, p_\infty)$ for any $p_n \rightarrow \infty$.

The Euler-Lagrange equation for the minimizer of $\mu(g, \tau)$ is

$$(3.5) \quad \tau(-4\Delta u + Ru) - u \log u^2 - nu = \mu(g, \tau)u.$$

For the general Ricci flow on the noncompact manifold we have the following result and the proof is almost identical with the compact case, see [31, Section 3.1],

Theorem 3.3. *If (M^n, g) is a manifold with bounded geometry such that a solution $g(t)$ of bounded curvature to the Ricci flow with $g(0) = g$ exists for $t \in [0, T)$, then for any $\bar{\tau} \in (0, T)$, $\mu(g, \bar{\tau}) < 0$ unless (M^n, g) is isometric to (\mathbb{R}^n, g_E) .*

Proof. Let $\tau(t) = \bar{\tau} - t$ and $y \in M$ and consider the corresponding fundamental solution

$$(3.6) \quad v(x, t) = (4\pi\tau(t))^{-n/2} e^{-f(x, t)}, \quad t \in [0, \bar{\tau})$$

to the adjoint heat equation

$$\frac{\partial v}{\partial t} = -\Delta_{g(t)} v + R_{g(t)} v$$

with $\lim_{t \nearrow \bar{\tau}} v(\cdot, t) = \delta_y$.

The existence of the fundamental solutions to the adjoint heat equation on non-compact manifold can be found in [11]. The basic properties of the fundamental solutions are proved in [14].

Then by the monotonicity of the entropy,

$$(3.7) \quad \mu(g, \bar{\tau}) = \mu(g, \tau(0)) \leq \mathcal{W}(g(0), f(0), \tau(0)) \leq \lim_{t \nearrow \bar{\tau}} \mathcal{W}(g(t), f(t), \tau(t)) = 0.$$

where the proof of the last limit in (3.7) can be found in [14, Theorem 7.1]. If $\mu(g, \bar{\tau}) = 0$, $\mathcal{W}(g(t), f(t), \tau(t)) = 0$ since it is monotone. Therefore from the formula

$$(3.8) \quad \frac{d\mathcal{W}(g(t), f(t), \tau(t))}{dt} = 2\tau \int_M \left| Ric + \nabla^2 f - \frac{g}{2\tau} \right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV$$

we have

$$(3.9) \quad Ric + \nabla^2 f - \frac{g}{2\tau} \equiv 0$$

for $t \in [0, \bar{\tau}]$, so that $g(t)$ is a shrinking soliton with singular time $\bar{\tau}$. So from

$$\tau(t) \max_M |Rm(g(t))| \equiv \text{const}$$

for $t \in [0, \bar{\tau}]$, we conclude that $|Rm(g(t))| \equiv 0$. In particular g is Ricci-flat and we have from (3.9)

$$(3.10) \quad \nabla^2 f - \frac{g}{2\bar{\tau}} \equiv 0.$$

Let $\bar{f} = 4\bar{\tau}f$, then $\nabla^2 \bar{f} = 2g$ and hence \bar{f} is a convex function.

Let O be a fixed point, then for any point $x \in M$ we have a minimal geodesic $s(t)$, $0 \leq t \leq d(x, O)$ such that $|\dot{s}(t)| = 1$. Then we have

$$(3.11) \quad \begin{aligned} \frac{d\bar{f}(s(t))}{dt} &= \langle \nabla \bar{f}, \nabla d \rangle = 0 \\ \frac{d^2 \bar{f}(s(t))}{dt^2} &= \nabla^2 \bar{f}(\nabla d, \nabla d) = 2g(\nabla d, \nabla d) = 2 \end{aligned}$$

So we have $\bar{f}(s(t)) = \bar{f}(O) + t\langle \nabla \bar{f}, \nabla d \rangle_{t=0} + t^2$. In other words, \bar{f} is quadratically increasing and therefore has a minimal point O_1 . Then we have $\bar{f}(x) = \bar{f}(O_1) + d^2(x, O_1)$. In particular, by taking trace of (3.10) we have

$$\Delta d^2 = 2n.$$

Therefore (M^n, g) is isometric to (\mathbb{R}^n, g_E) by the comparison theorem since g is Ricci-flat. \square

Now we have the following crucial result.

Theorem 3.4. *If (M^n, g) is an AE manifold such that the scalar curvature $R > 0$, then $\lim_{\tau \rightarrow \infty} \mu(g, \tau) = 0$.*

Proof. If the conclusion does not hold, we can find a sequence $(\tau_k)_{k \in \mathbb{N}}$ going to ∞ and $\lim_{k \rightarrow \infty} \mu(g, \tau_k) = \mu_\infty$, so that either μ_∞ is a finite negative number or $\mu_\infty = -\infty$.

By the above assumption, $\mu(g, \tau_k)$ has a minimizer u_k and it satisfies

$$(3.12) \quad \tau_k(-4\Delta u_k + Ru_k) - u_k \log u_k^2 - nu_k = \mu(g, \tau_k)u_k$$

and

$$(3.13) \quad \int_M u_k^2 (4\pi\tau_k)^{-\frac{n}{2}} dV = 1.$$

Claim 1. u_k are uniformly bounded.

We first prove a lemma.

Lemma 3.5. *For $u \in W^{1,2}(M)$, the following Sobolev inequality holds*

$$(3.14) \quad \left(\int_M u^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} \leq C \int_M 4|\nabla u|^2 + Ru^2 dV.$$

Proof. Let $M^n = K \sqcup E$ be the disjoint union of a compact set K and AE end E and K_1 a compact set such that $K \subset \subset K_1$. Choose a partition of unity $1 = \phi_0 + \phi_1$ such that $\text{supp } \phi_1 \subset E$ and $\phi_0 = 1$ on K and $\text{supp } \phi_0 \subset K_1$.

For any $u \in W^{1,2}(M)$, we have

$$\|u\|_{\frac{2n}{n-2}} = \left\| \sum \phi_i u \right\|_{\frac{2n}{n-2}} \leq \sum \|\phi_i u\|_{\frac{2n}{n-2}}.$$

By the L^2 Sobolev inequality on manifold with bounded geometry, see [2],

$$\begin{aligned} \left(\int_M (\phi_0 u)^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} &\leq C \int_M |\nabla(\phi_0 u)|^2 + (\phi_0 u)^2 dV \\ &\leq C \int_{K_1} |\nabla \phi_0 u|^2 + |\phi_0 \nabla u|^2 + (\phi_0 u)^2 dV \\ &\leq C \int_{K_1} |\nabla u|^2 + u^2 dV \\ (3.15) \quad &\leq C \int_{K_1} 4|\nabla u|^2 + Ru^2 dV. \end{aligned}$$

The last inequality holds since we assume $R > 0$.

On the AE end E , by enlarging K and K_1 if necessary, we can assume the L^2 Sobolev inequality of the Euclidean type holds. To be precise, on \mathbb{R}^n we have the L^2 Sobolev inequality [1]:

$$(3.16) \quad \left(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \leq C \int_{\mathbb{R}^n} |\nabla_{g_E} u|^2 dx$$

for any $u \in C_0^1(\mathbb{R}^n)$ and some constant $C > 0$.

Since E is the AE end, by shrinking it if necessary, we can assume that there exists a small $\epsilon > 0$ such that

$$(1 - \epsilon)dx \leq dV \leq (1 + \epsilon)dx$$

$$(1 - \epsilon)|\nabla_{g_E} u|^2 \leq |\nabla u|^2 \leq (1 + \epsilon)|\nabla_{g_E} u|^2.$$

Hence, for any $u \in C_0^1(E)$, there is a $C > 0$ such that

$$(3.17) \quad \begin{aligned} \left(\int_E u^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} &\leq \left((1 + \epsilon) \int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \\ &\leq C(1 + \epsilon)^{\frac{n-2}{n}} \int_{\mathbb{R}^n} |\nabla_{g_E} u|^2 dx \\ &\leq C(1 + \epsilon)^{\frac{n-2}{n}} \left(\frac{1}{1 - \epsilon} \right) \int_{\mathbb{R}^n} |\nabla u|^2 dx \\ &\leq C(1 + \epsilon)^{\frac{n-2}{n}} \left(\frac{1}{1 - \epsilon} \right)^2 (1 + \epsilon) \int_{\mathbb{R}^n} |\nabla u|^2 dV \\ &\leq C \int_E |\nabla u|^2 dV. \end{aligned}$$

So we have

$$(3.18) \quad \begin{aligned} \left(\int_M (\phi_1 u)^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} &\leq C \int_M |\nabla(\phi_1 u)|^2 dV \\ &\leq C \int_M |\nabla \phi_1 u|^2 + |\phi_1 \nabla u|^2 dV \\ &\leq C \left(\int_M |\nabla u|^2 dV + \int_{K_1} u^2 dV \right) \\ &\leq C \int_M 4|\nabla u|^2 + Ru^2 dV. \end{aligned}$$

Combining (3.15) and (3.18), (3.14) holds. \square

We can now prove the claim by using the Moser iteration. This is known to experts but we write it down for the convenience of readers. For the sake of simplicity, we will not write down the subscript k explicitly throughout and set $\mu = \mu(g, \tau_k)$.

Proof of Claim 1, see also [43, Lemma 2.1]. From (3.12) we have

$$4\Delta u - Ru + \frac{2}{\tau} u \log u + \frac{n + \mu}{\tau} u \geq 0.$$

Since $\mu \leq 0$, we have

$$(3.19) \quad 4\Delta u - Ru + \frac{2}{\tau}u \log u + \frac{n}{\tau}u \geq 0.$$

Through computation, for $p \geq 1$

$$(3.20) \quad \begin{aligned} 4\Delta u^p &= 4p(p-1)u^{p-2}|\nabla u|^2 + 4pu^{p-1}\Delta u \\ &\geq 4pu^{p-1}\Delta u \\ &\geq -\frac{2p}{\tau}u^p \log u - \frac{np}{\tau}u^p + pRu^p. \end{aligned}$$

We set $w = u^p$ and ϕ to be a test function. From (3.20) we have

$$4 \int \langle \nabla(w\phi^2), \nabla w \rangle dV \leq \frac{2p}{\tau} \int w^2 \phi^2 \log u dV + \frac{np}{\tau} \int w^2 \phi^2 dV - \int pRw^2 \phi^2 dV.$$

On the other hand, since

$$\langle \nabla(w\phi^2), \nabla w \rangle = |\nabla(w\phi)|^2 - |\nabla\phi|^2 w^2$$

we have

$$(3.21) \quad 4 \int |\nabla(w\phi)|^2 dV \leq 4 \int |\nabla\phi|^2 w^2 dV + \frac{2p}{\tau} \int w^2 \phi^2 \log u dV + \left(\frac{np}{\tau} - pR\right) \int w^2 \phi^2 dV.$$

There is a constant $c_1 > 0$ such that

$$\log u \leq u^{\frac{2}{n}} + c_1.$$

Hence

$$(3.22) \quad \begin{aligned} \frac{2p}{\tau} \int w^2 \phi^2 \log u dV &\leq \frac{2p}{\tau} \int w^2 \phi^2 u^{\frac{2}{n}} dV + \frac{2c_1 p}{\tau} \int w^2 \phi^2 dV \\ &\leq \frac{2p}{\tau} \left(\int (w\phi)^{\frac{2n}{n-1}} dV \right)^{\frac{n-1}{n}} \left(\int u^2 dV \right)^{\frac{1}{n}} + \frac{2c_1 p}{\tau} \int w^2 \phi^2 dV \\ &= \frac{\sqrt{4\pi} 2p}{\sqrt{\tau}} \left(\int (w\phi)^{\frac{2n}{n-1}} dV \right)^{\frac{n-1}{n}} + \frac{2c_1 p}{\tau} \int w^2 \phi^2 dV \end{aligned}$$

since (3.13) holds.

From Hölder's inequality,

$$(3.23) \quad \begin{aligned} \left(\int (w\phi)^{\frac{2n}{n-1}} dV \right)^{\frac{n-1}{n}} &\leq \left(\int (w\phi)^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{2n}} \left(\int w^2 \phi^2 dV \right)^{\frac{1}{2}} \\ &\leq \lambda \left(\int (w\phi)^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} + \frac{1}{4\lambda} \int w^2 \phi^2 dV, \end{aligned}$$

where the last line is from Young's inequality for any $\lambda > 0$.

So from (3.22),

$$(3.24) \quad \begin{aligned} \frac{2p}{\tau} \int w^2 \phi^2 \log u \, dV &\leq \frac{c_2 \lambda p}{\sqrt{\tau}} \left(\int (w\phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \\ &+ \frac{c_2 p}{4\lambda \sqrt{\tau}} \int w^2 \phi^2 \, dV + \frac{2c_1 p}{\tau} \int w^2 \phi^2 \, dV. \end{aligned}$$

From lemma (3.5), (3.21) (3.24), we have

$$(3.25) \quad \begin{aligned} \frac{1}{C} \left(\int (w\phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} &\leq \int 4|\nabla(w\phi)|^2 + R(w\phi)^2 \, dV \\ &\leq 4 \int |\nabla \phi|^2 w^2 \, dV + \frac{2p}{\tau} \int w^2 \phi^2 \log u \, dV + \frac{np}{\tau} \int w^2 \phi^2 \, dV \\ &\leq 4 \int |\nabla \phi|^2 w^2 \, dV + \frac{c_2 \lambda p}{\sqrt{\tau}} \left(\int (w\phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \\ &\quad + \frac{c_2 p}{4\lambda \sqrt{\tau}} \int w^2 \phi^2 \, dV + \frac{2c_1 p}{\tau} \int w^2 \phi^2 \, dV \\ &\quad + \frac{np}{\tau} \int w^2 \phi^2 \, dV. \end{aligned}$$

If we choose λ satisfies $\frac{c_2 \lambda p}{\sqrt{\tau}} = \frac{1}{2C}$, that is, $\lambda = \frac{\sqrt{\tau}}{2C c_2 p}$, then from (3.25), there exists a $C_0 > 0$ such that

$$(3.26) \quad \left(\int (w\phi)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq C_0 \int |\nabla \phi|^2 w^2 \, dV + \frac{C_0 p^2}{\tau} \int w^2 \phi^2 \, dV.$$

For any point x on M , we choose ϕ_k such that it is supported on $B(x, \sqrt{\tau}(1 + 1/2^k))$ and $\phi_k = 1$ on $B(x, \sqrt{\tau}(1 + 1/2^{k+1}))$ such that $|\nabla \phi_k| \leq \frac{C 2^k}{\sqrt{\tau}}$.

From (3.26) we have

$$(3.27) \quad \begin{aligned} \left(\int_{B(x, \sqrt{\tau}(1+1/2^{k+1}))} w^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} &\leq \left(\int (w\phi_k)^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \\ &\leq C_0 \int |\nabla \phi_k|^2 w^2 \, dV + \frac{C_0 p^2}{\tau} \int w^2 \phi_k^2 \, dV \\ &\leq \frac{C_1 2^{2k} p^2}{\tau} \int_{B(x, \sqrt{\tau}(1+1/2^k))} w^2 \, dV. \end{aligned}$$

If we set $p_0 = \frac{n}{n-2}$ and choose $p = p_0^k$, from (3.27) we have

$$(3.28) \quad \left(\int_{B(x, \sqrt{\tau}(1+1/2^{k+1}))} u^{2p_0^{k+1}} \, dV \right)^{\frac{n-2}{n}} \leq \frac{C_1 (2p_0)^{2k}}{\tau} \int_{B(x, \sqrt{\tau}(1+1/2^k))} w^{2p_0^k} \, dV,$$

or equivalently,

(3.29)

$$\left(\int_{B(x, \sqrt{\tau}(1+1/2^{k+1}))} u^{2p_0^{k+1}} dV \right)^{\frac{1}{p_0^{k+1}}} \leq \frac{C_1^{\frac{1}{p_0^k}} (2p_0)^{\frac{2k}{p_0^k}}}{\tau^{\frac{1}{p_0^k}}} \left(\int_{B(x, \sqrt{\tau}(1+1/2^k))} u^{2p_0^k} dV \right)^{\frac{1}{p_0^k}}.$$

Let $k = 0, 1, \dots$, and by iteration,

$$(3.30) \quad \max_{B(x, \sqrt{\tau})} u^2 \leq \frac{C_1^{\sum_{k \geq 0} \frac{1}{p_0^k}} p_0^{\sum_{k \geq 0} \frac{2k}{p_0^k}}}{\tau^{\sum_{k \geq 0} \frac{1}{p_0^k}}} \left(\int_{B(x, 2\sqrt{\tau})} u^2 dV \right) \leq \frac{C_2}{\tau^{\frac{n}{2}}} \left(\int_{B(x, 2\sqrt{\tau})} u^2 dV \right)$$

since $\sum_{k \geq 0} \frac{1}{p_0^k} = \frac{n}{2}$ and $\sum_{k \geq 0} \frac{2k}{p_0^k}$ converges. As

$$\int_{B(x, 2\sqrt{\tau})} u^2 dV \leq \int_M u^2 dV = (4\pi\tau)^{\frac{n}{2}},$$

we conclude from (3.30) that

$$\max_M u^2 \leq C_3$$

for some constant $C_3 > 0$.

On the other hand, $u_k > 0$ by the strong maximum principle, see [34]. Hence all u_k are uniformly bounded.

Since every minimizer is exponentially decaying, see [43, Lemma 2.3], there is a maximum point p_k for u_k . Since $\Delta u_k(p_k) \leq 0$, at p_k we have in (3.12)

$$\tau_k R u_k - u_k \log u_k^2 - n u_k - \mu_k u_k \leq 0.$$

As $u_k > 0$, we have

$$u_k(p_k) \geq \exp \left(\frac{R(p_k) \tau_k - n - \mu_k}{2} \right) \geq \exp \left(\frac{-n - \mu_k}{2} \right).$$

As we have proved that u_k is uniformly bounded, μ_k cannot tend to $-\infty$. In other words, μ_∞ is finite.

From (3.12) we have

$$\int_K u_k^2 dV + \int_E u_k^2 dV = (4\pi\tau_k)^{\frac{n}{2}}.$$

Since u_k are bounded and K has finite volume, there is a $c_0 \in (0, 1]$ satisfying

$$(3.31) \quad \int_E u_k^2 dV \geq c_0 (4\pi\tau_k)^{\frac{n}{2}}.$$

We define functions $\tilde{u}_k(x) = u_k(\sqrt{\tau_k}x)$, a new metric on E as $\tilde{g}_{ij}(x) = g_{ij}(\sqrt{\tau_k}x)$, the corresponding Laplace operator $\tilde{\Delta}_k = \frac{1}{\sqrt{\det \tilde{g}}} \partial_i \sqrt{\det \tilde{g}} \tilde{g}^{ij} \partial_j$ and scalar curvature

$$\tilde{R}(x) = \frac{1}{\tau_k} R(\sqrt{\tau_k}x).$$

The metric \tilde{g} on E , after a diffeomorphism, is nothing but $\tau_k^{-1}g$. So by the AE condition, (E, \tilde{g}) converges in the Cheeger-Gromov sense to $(\mathbb{R}^n \setminus \{0\}, g_E)$ and the convergence is smooth away from the origin.

So (3.12) becomes

$$(3.32) \quad -4\tilde{\Delta}_k \tilde{u}_k + \tilde{R}\tilde{u}_k - \tilde{u}_k \log \tilde{u}_k^2 - n\tilde{u}_k = \mu_k \tilde{u}_k$$

All \tilde{u}_k can be regarded as functions defined on \mathbb{R}^n except for a ball with center 0. We next prove that there is a limit in $W^{1,2}(\mathbb{R}^n)$ for the sequence $\{\tilde{u}_k\}$.

Since $\{\mu_k\}$ are bounded, from (3.12) and (3.13) we have, for details see [39],

$$(3.33) \quad \tau_k \int_M |\nabla u_k|^2 (4\pi\tau_k)^{-\frac{n}{2}} dV \leq C.$$

Therefore, for any annulus $C_{a,A} = \{x \in \mathbb{R}^n \mid a < |x| < A\}$, we have a uniform constant $C_1 > 0$ such that

$$\int_{C_{a,A}} \tilde{u}_k^2 d\tilde{V} \leq C_1,$$

and

$$\int_{C_{a,A}} |\tilde{\nabla} \tilde{u}_k|^2 d\tilde{V} \leq C_1.$$

In other words, \tilde{u}_k are bounded in $W^{1,2}(C_{a,A})$ and hence a subsequence of $\{\tilde{u}_k\}$ converges weakly to a function u_∞ in $W^{1,2}(C_{a,A})$ and by Sobolev imbedding converges strongly to u_∞ in $L^p(C_{a,A})$ if $1 \leq p < 2n/n-2$. Choosing two sequences $a_m \rightarrow 0$ and $A_m \rightarrow \infty$ for $m = 1, 2, \dots$, by the diagonal argument replacing $\{\tilde{u}_k\}$ by a subsequence if necessary, we have a function u_∞ defined on $\mathbb{R}^n \setminus \{0\}$ such that for every compact set C in $\mathbb{R}^n \setminus \{0\}$, there is an $N > 0$ such that $\{\tilde{u}_k, k \geq N\}$ converges weakly to u_∞ in $W^{1,2}(\mathbb{R}^n \setminus \{0\})$ and strongly in $L^p(\mathbb{R}^n \setminus \{0\})$ if $1 \leq p < 2n/n-2$.

Moreover the convergence can be made in $C_{\text{loc}}^\alpha(\mathbb{R}^n \setminus \{0\})$ for some $\alpha > 0$, as proved in [39]. Therefore if $k \rightarrow \infty$ in (3.32), we have

$$(3.34) \quad -4\Delta_E u_\infty - u_\infty \log u_\infty^2 - n u_\infty = \mu_\infty u_\infty$$

where Δ_E means the standard Laplacian operator on \mathbb{R}^n . By the standard regularity property of elliptic operator, we know that $u_\infty \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and either $u_\infty \equiv 0$ or $u_\infty > 0$ by the strong maximum principle [34].

Moreover we have

$$(3.35) \quad \int_{\mathbb{R}^n \setminus \{0\}} u_\infty^2 dx \leq (4\pi)^{\frac{n}{2}},$$

and there exists a $C > 0$ such that

$$(3.36) \quad \int_{\mathbb{R}^n \setminus \{0\}} |\nabla u_\infty|^2 dx \leq C.$$

Claim 2. $u_\infty \in W^{1,2}(\mathbb{R}^n)$.

Proof of Claim 2. We first prove a lemma.

Lemma 3.6. *For a function $f \in C^1(\mathbb{R}^n \setminus \{0\})$, if $|f(x)| \leq C|x|^{-\alpha}$ for some $\alpha < n-1$ and small x and $|\nabla f|$ is integrable on the punctured ball $B(0, 1) \setminus \{0\}$, then the function*

$$\tilde{f}(x) = \begin{cases} f(x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

has the weak derivative

$$g_i(x) = \begin{cases} \partial_i f(x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

for $i = 1, 2, \dots, n$

Proof. For any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \int_{\mathbb{R}^n} \tilde{f} \partial_i \phi \, dx &= \lim_{r \rightarrow 0} \int_{\mathbb{R}^n \setminus B(0, r)} f \partial_i \phi \, dx \\ &= - \lim_{r \rightarrow 0} \int_{\mathbb{R}^n \setminus B(0, r)} \partial_i f \phi \, dx + \lim_{r \rightarrow 0} \int_{S(0, r)} f \phi v^i \, d\sigma \\ &= - \int_{\mathbb{R}^n} g_i \phi \, dx + \lim_{r \rightarrow 0} \int_{S(0, r)} f \phi v^i \, d\sigma \end{aligned}$$

where v^i is the i th component of the inner normal vector of $S(0, r)$. The first integral in the last line is finite since g_i is integrable by our assumption.

From the condition,

$$\left| \int_{S(0, r)} f \phi v^i \, d\sigma \right| \leq C' r^{n-1} \max_{x \in S(0, r)} |f| \leq C' C r^{n-1-\alpha}.$$

Since $\alpha < n-1$ we conclude that

$$\lim_{r \rightarrow 0} \int_{S(0, r)} f \phi v^i \, d\sigma = 0$$

and the lemma follows. \square

Applying Moser's iteration to (3.34) as the proof of Claim 1, we have for any $0 < r \leq 1$ and $|p| = r$,

$$\max_{B(p, r/4)} u_\infty^2 \leq \frac{C}{r^n} \int_{B(p, r/2)} u_\infty^2 \, dx \leq \frac{C'}{r^n}.$$

Hence we have

$$u_\infty(x) \leq \frac{C}{|x|^{n/2}}$$

for $|x| \leq 1$. Therefore, by combining (3.36) we can apply Lemma 3.6 to conclude that u_∞ can be extended to \mathbb{R}^n . Moreover from (3.35) and (3.36), $u_\infty \in W^{1,2}(\mathbb{R}^n)$.

Case 1: $u_\infty > 0$.

From (3.35) we have

$$0 < \int_{\mathbb{R}^n} u_\infty^2 (4\pi)^{-\frac{n}{2}} dx = c_1^2 \leq 1.$$

So if we set $\tilde{u}_\infty = u_\infty/c_1$, from (3.34) we have

$$\begin{aligned} & \int_{\mathbb{R}^n} (4|\nabla \tilde{u}_\infty|^2 - \tilde{u}_\infty^2 \log \tilde{u}_\infty^2 - n\tilde{u}_\infty^2) (4\pi)^{-\frac{n}{2}} dx \\ &= \frac{1}{c_1^2} \int_{\mathbb{R}^n} (4|\nabla u_\infty|^2 - u_\infty^2 \log u_\infty^2 - nu_\infty^2) (4\pi)^{-\frac{n}{2}} dx + \log c_1^2 \\ (3.37) \quad &= \mu_\infty + \log c_1^2 < 0 \end{aligned}$$

since $\mu_\infty < 0$ and $c_1^2 < 1$. But it contradicts the fact that $\mu_{\mathbb{R}^n}(g_E, 1) = 0$.

Case 2: $u_\infty \equiv 0$.

In this case it means that $\tilde{u}_k(x) = u_k(\sqrt{\tau_k}x)$ converges uniformly to 0 on any compact set of E .

We can assume that

$$\limsup_{k \rightarrow \infty} \max_{x \in \mathbb{R}^n \setminus B(0,1)} \tilde{u}_k(x) = 0.$$

Otherwise, if there exists a sequence $(p_k)_{k \in \mathbb{N}}$ such that $\tilde{u}_k(p_k) \geq c > 0$, by our assumption $p_k \rightarrow \infty$. On the other hand, (M, \tilde{g}_k, p_k) converges smoothly to $(\mathbb{R}^n, g_E, p_\infty)$ and hence $\tilde{u}_k(x)$ converges to u'_∞ which is not identically zero. Then like case 1, we have a contradiction.

Choose a small constant $a > 0$ such that

$$(3.38) \quad \int_{E \setminus B(0, a\sqrt{\tau_k})} u_k^2 dV \geq \frac{c_0}{2} (4\pi\tau_k)^{\frac{n}{2}}.$$

This is possible since (3.31) holds and u_k are uniformly bounded.

Choose a function ϕ such that $\phi \in C_0^\infty(\mathbb{R}^n \setminus B(0, a))$ and $\phi = 1$ on $\mathbb{R}^n \setminus B(0, 2a)$. Then we have, like (3.21)

$$\begin{aligned} & \int 4|\tilde{\nabla}(\phi \tilde{u}_k)|^2 + (\tilde{R} - n)(\phi \tilde{u}_k)^2 - (\phi \tilde{u}_k)^2 \log(\tilde{u}_k^2) d\tilde{V} \\ &= \int 4|\tilde{\nabla}(\phi \tilde{u}_k)|^2 + (\tilde{R} - n)(\phi \tilde{u}_k)^2 - (\phi \tilde{u}_k)^2 \log(\phi \tilde{u}_k)^2 d\tilde{V} + \int (\phi \tilde{u}_k)^2 \log(\phi^2) d\tilde{V} \\ (3.39) \quad & \leq \int 4|\tilde{\nabla} \phi|^2 \tilde{u}_k^2 d\tilde{V} + \mu_k \int (\phi \tilde{u}_k)^2 d\tilde{V} \leq C \int_{C_{a,2a}} \tilde{u}_k^2 d\tilde{V} + \mu_k \int (\phi \tilde{u}_k)^2 d\tilde{V}. \end{aligned}$$

That is,

$$\begin{aligned} & \int 4|\tilde{\nabla}(\phi \tilde{u}_k)|^2 + (\tilde{R} - n)(\phi \tilde{u}_k)^2 - (\phi \tilde{u}_k)^2 \log(\phi \tilde{u}_k)^2 d\tilde{V} \\ (3.40) \quad & \leq C_1 \int_{C_{a,2a}} \tilde{u}_k^2 d\tilde{V} + \mu_k \int (\phi \tilde{u}_k)^2 d\tilde{V}. \end{aligned}$$

But from our assumption \tilde{u}_k converges to 0 uniformly on $C_{a,2a}$, there is an $\epsilon > 0$ such that for large k ,

$$(3.41) \quad \int 4|\tilde{\nabla}(\phi\tilde{u}_k)|^2 + (\tilde{R} - n)(\phi\tilde{u}_k)^2 - (\phi\tilde{u}_k)^2 \log(\phi\tilde{u}_k)^2 d\tilde{V} \leq \epsilon + \mu_k \int (\phi\tilde{u}_k)^2 d\tilde{V}.$$

On the other hand,

$$(4\pi)^{\frac{n}{2}} \geq \int \tilde{u}_k^2 d\tilde{V} \geq \int (\phi\tilde{u}_k)^2 d\tilde{V} \geq \int_{\mathbb{R}^n \setminus B(0,2a)} \tilde{u}_k^2 d\tilde{V} \geq \left(\frac{c_0}{2} - \epsilon\right)(4\pi)^{\frac{n}{2}}.$$

So if we set

$$\int (\phi\tilde{u}_k)^2 d\tilde{V} = c_2^2 (4\pi)^{\frac{n}{2}},$$

and $\psi_k = \frac{\phi\tilde{u}_k}{c_2}$, where $c_2 \in [\frac{c_0}{2} - \epsilon, 1]$, from (3.41) we have,

$$(3.42) \quad \begin{aligned} & \int \left(4|\tilde{\nabla}\psi_k|^2 + (\tilde{R} - n)(\psi_k)^2 - (\psi_k)^2 \log(\psi_k)^2 \right) (4\pi)^{-\frac{n}{2}} d\tilde{V} \\ & \leq c_2^2 \epsilon + \mu_k + \log c_2^2 \leq c_2^2 \epsilon + \mu_k. \end{aligned}$$

When k is large, ϵ can be chosen to be small such that $c_2^2 \epsilon + \mu_k < \mu_k/2$.

Since \tilde{g}_k uniformly converges to g_E on $\mathbb{R}^n \setminus B(0, a)$ and $\psi_k^2 \log \psi_k^2 \leq 0$ by our assumption $\tilde{u}_k \leq c_2$ for large k , we have

$$\int 4|\nabla_E \psi_k|^2 - n\psi_k^2 - \psi_k^2 \log \psi_k^2 dV_E < \mu_k/3$$

for k large enough and

$$\lim_{k \rightarrow \infty} \int \psi_k^2 dV_E = 1.$$

But this contradicts the fact that $\mu_{\mathbb{R}^n \setminus B(0,a)}(g_E, 1) = 0$ on $\mathbb{R}^n \setminus B(0, a)$.

Thus we have finished the proof of Theorem 3.4. \square

Following the proof of Theorem 3.4, we have the following uniform version.

Theorem 3.7. *Let (M_i^n, g_i) be a family of AE manifolds of the same order $\sigma > 0$ with positive scalar curvature. For some compact sets $K_i \subset M_i^n$, we have a family of diffeomorphisms $\Phi_i : M_i^n \setminus K_i \rightarrow \mathbb{R}^n \setminus B(0, R)$ such that under these identifications,*

$$(3.43) \quad |g_{ij} - \delta_{ij}| \leq C_0 r^{-\sigma_i}, \quad |\partial^{[k]} g_{ij}| \leq C_k r^{-\sigma-k}.$$

for some constants $C_k, k = 0, 1, \dots$. Moreover, there exist compact sets K'_i containing K_i such that $\text{dis}_{g_E}(K_i, K'_i) \geq d_0$ and

$$\left(\int_{M_i - K'_i} u^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} \leq C \int_{M_i - K'_i} |\nabla u|^2 dV$$

for some $d_0 > 0$, $C > 0$ and any $u \in C_0^1(M_i - K'_i)$. In addition, if $|Rm|_{g_i} \leq R_0$, $\text{inj}_{g_i} \geq i_0$, $\text{Vol}_{g_i}(K'_i) \leq V_0$ and $\inf_{p \in K'_i} R_{g_i}(p) \geq r_0$ for some positive constants R_0, r_0, i_0 and V_0 , we have

$$\lim_{\tau \rightarrow +\infty} \mu_{M_i}(g_i, \tau) = 0$$

for all g_i uniformly.

Remark 3.8. We can get a uniform constant for Lemma 3.5 since the Sobolev constant only depends on the bounds of curvature and injective radius. The volume control of K'_i is used to prove (3.31).

Next, we use Theorem 3.4 to prove the no local collapsing theorem in the case of AE manifold. Recall that a Riemannian manifold is κ -noncollapsed on all scales if for any metric ball $B(x, r)$ satisfying $|Rm| \leq r^{-2}$ for all $y \in B(x, r)$, we have

$$\frac{\text{Vol}B(x, r)}{r^n} \geq \kappa.$$

Following the celebrated work of Perelman, we have

Theorem 3.9. *Let $g(t)$, $t \in [0, \infty)$, be the Ricci flow solution on an AE manifold M^n with $R > 0$, then there exists $\kappa > 0$ such that $g(t)$ is κ -noncollapsed on all scales.*

Proof. Since Ricci flow preserves the AE condition. So there exists a $\kappa_1 > 0$ such that for any $t \in [0, 1]$, $r > 0$, we have

$$(3.44) \quad \frac{\text{Vol}B_{g(t)}(x, r)}{r^n} \geq \kappa_1,$$

where $B_{g(t)}(x, r)$ is a metric ball in $(M^n, g(t))$.

For $t \in [1, \infty)$, $r > 0$ and $p \in M$ such that $|Rm| \leq r^{-2}$ in $B_{g(t)}(x, r)$ we have the following inequality whose proof can be found in [12, Proposition 5.37]

$$(3.45) \quad \mu(g(t), r^2) \leq \log \frac{\text{Vol}B_{g(t)}(x, r)}{r^n} + C(n).$$

Then by (3.45), Theorem 3.4 and the continuity and monotonicity of $\mu(g, \tau)$, there exists a constant C such that

$$C \leq \mu(g(0), r^2 + t) \leq \mu(g(t), r^2) \leq \log \frac{\text{Vol}B_{g(t)}(x, r)}{r^n} + C(n).$$

We conclude that there exists $\kappa_2 > 0$ such that

$$(3.46) \quad \frac{\text{Vol}B_{g(t)}(x, r)}{r^n} \geq \kappa_2.$$

Combining (3.44) and (3.46), we can find $\kappa = \min(\kappa_1, \kappa_2) > 0$ such that $g(t)$ is κ -noncollapsed on all scales. \square

4. ANALYSIS OF SINGULARITY AT TIME INFINITY

For the Ricci flow $(M, g(t))$, $t \in [0, \infty)$, there are two different types of singularity at infinity classified by Hamilton, see [20].

Case 1 (Type IIb): $\sup_{M \times [0, \infty)} t |\text{Rm}| = \infty$.

In this case, we take any sequences of times $T_i \rightarrow \infty$ and then choose $(x_i, t_i) \in M^n \times [0, T_i]$ such that

$$t_i(T_i - t_i) |\text{Rm}|(x_i, t_i) = \sup_{M^n \times (0, T_i]} t(T_i - t) |\text{Rm}|(x, t).$$

If we set $Q_i = |\text{Rm}|(x_i, t_i)$, it can be proved that $g_i(t) = Q_i g(t_i + Q_i^{-1}t)$ converges smoothly in the Cheeger-Gromov sense to a complete eternal Ricci flow solution $(M_\infty, g_\infty(t), x_\infty)$, $t \in (-\infty, +\infty)$.

Then for any $\tau > 0$,

$$\begin{aligned} \mu(g_\infty(0), \tau) &\geq \limsup_{i \rightarrow \infty} \mu(Q_i g(t_i), \tau) \\ &\geq \limsup_{i \rightarrow \infty} \mu(g(t_i), \frac{\tau}{Q_i}) \\ (4.1) \quad &\geq \limsup_{i \rightarrow \infty} \mu(g(0), \frac{\tau}{Q_i} + t_i) = 0 \end{aligned}$$

where the first inequality follows from Lemma 3.2, the last from the monotonicity of μ and the equality is from Theorem 3.4.

From Theorem 3.3, it must be the case that M^n is isometric to \mathbb{R}^n . But this is impossible since $|\text{Rm}|_{g_\infty(0)}(x_\infty) = \lim_{i \rightarrow \infty} |\text{Rm}|_{g_i(0)}(x_i) = 1$.

Case 2 (Type III): $\sup_{M \times [0, \infty)} t |\text{Rm}| < \infty$.

In this case, suppose (x_i, t_i) is a sequence of points and times with $t_i \rightarrow \infty$ and

$$t_i |\text{Rm}|(x_i, t_i) = t_i \sup_{x \in M} |\text{Rm}|(x, t_i) \geq c$$

for some $c > 0$. Then like the first case $g_i(t) = Q_i g(t_i + Q_i^{-1}t)$, $t \in [-t_i Q_i, \infty)$, converges to $(M_\infty, g_\infty(t), x_\infty)$, $t \in (-c, +\infty)$, which again we derive a contradiction.

Therefore, we have proved that the singularity at infinity is of type III, and

$$(4.2) \quad \lim_{t \rightarrow \infty} t \sup_M |\text{Rm}(t)| = 0.$$

We choose an $\epsilon \in (0, 1)$ to be determined later. From (4.2) we can assume for t large enough we have

$$(4.3) \quad \sup_M |\text{Rm}| \leq \frac{\epsilon}{1+t}.$$

So by a translation of time, we can assume (4.3) holds for any $t \geq 0$.

Next, we prove a gradient estimate and Harnack inequality for the solution of heat equation under the condition of (4.3). The proof is a long time version of the Li-Yau estimates, see [25].

Set $u_0 = r^{-2-\sigma}$ and we consider the positive solution u of the heat equation

$$(4.4) \quad u_t = \Delta u$$

with the initial condition $u(0) = u_0$.

It can be proved by using the maximum principle as in the proof of Theorem 2.2, that for any $T > 0$, $t \in [0, T]$, $u(t)$ has the same decaying condition as $u(0)$. To be precise, there exist $c_1(T) > 0$ and $c_2(T) > 0$ such that

$$(4.5) \quad c_1(T)r^{-2-\sigma} \leq u(t) \leq c_2(T)r^{-2-\sigma}.$$

Let $f = \log u$. The f satisfies

$$f_t = \Delta f + |\nabla f|^2.$$

We now prove the following lemma

Lemma 4.1. *Let $H(x, t) = t(|\nabla f|^2 - 2f_t)$. Then*

$$(4.6) \quad \Delta H - H_t \geq -2\nabla f \cdot \nabla H + \frac{t}{n}(|\nabla f|^2 - f_t)^2 - (|\nabla f|^2 - 2f_t) - 3|\nabla f|^2 - \frac{4\epsilon^2}{1+t}$$

(4.7)

Proof. We have

$$(4.8) \quad \Delta H = t\Delta(|\nabla f|^2 - 2f_t).$$

By using the Bochner's formula

$$(4.9) \quad \begin{aligned} \Delta|\nabla f|^2 &= 2|\nabla^2 f|^2 + 2Rc(\nabla f, \nabla f) + 2\langle \nabla \Delta f, \nabla f \rangle \\ &= 2|\nabla^2 f|^2 + 2Rc(\nabla f, \nabla f) - 2\langle \nabla(|\nabla f|^2 - f_t), \nabla f \rangle \\ &\geq 2|\nabla^2 f|^2 - \frac{2}{1+t}|\nabla f|^2 - 2\langle \nabla(|\nabla f|^2 - f_t), \nabla f \rangle \end{aligned}$$

where the last inequality follows from our curvature estimate.

On the other hand,

$$\Delta f_t = (\Delta f)_t - 2R_{ij}f_{ij} \leq (\Delta f)_t + 2|Ric|^2 + \frac{1}{2}|\nabla^2 f|^2.$$

So we get

$$(4.10) \quad \begin{aligned} \Delta H &\geq t \left(|\nabla^2 f|^2 - 2\langle \nabla(|\nabla f|^2 - f_t), \nabla f \rangle - 2(\Delta f)_t - \frac{2}{1+t}|\nabla f|^2 - 4|Ric|^2 \right) \\ &\geq \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2t\langle \nabla(|\nabla f|^2 - f_t), \nabla f \rangle \\ &\quad + 2t(|\nabla f|^2 - f_t)_t - 2|\nabla f|^2 - \frac{4\epsilon^2}{1+t}. \end{aligned}$$

Then we have

$$H_t = |\nabla f|^2 - 2f_t + t(|\nabla f|^2 - 2f_t)_t.$$

Therefore,

$$\begin{aligned}
\Delta H - H_t &\geq \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2t\langle \nabla(|\nabla f|^2 - f_t), \nabla f \rangle \\
&\quad + 2t(|\nabla f|^2 - f_t)_t - t(|\nabla f|^2 - 2f_t)_t - (|\nabla f|^2 - 2f_t) - 2|\nabla f|^2 - \frac{4\epsilon^2}{1+t} \\
&= \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2t\langle \nabla(|\nabla f|^2 - f_t), \nabla f \rangle \\
&\quad + t|\nabla f|_t^2 - (|\nabla f|^2 - 2f_t) - 2|\nabla f|^2 - \frac{4\epsilon^2}{1+t} \\
&= \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2t\langle \nabla(|\nabla f|^2 - f_t), \nabla f \rangle \\
&\quad + 2t\langle \nabla f_t, \nabla f \rangle + 2tRic(\nabla f, \nabla f) - (|\nabla f|^2 - 2f_t) - 2|\nabla f|^2 - \frac{4\epsilon^2}{1+t} \\
&\geq \frac{t}{n}(|\nabla f|^2 - f_t)^2 - 2\langle \nabla H, \nabla f \rangle - (|\nabla f|^2 - 2f_t) - 3|\nabla f|^2 - \frac{4\epsilon^2}{1+t}.
\end{aligned}
\tag{4.11}$$

□

Now we can use the above equation to derive the Li-Yau inequality, see [38][Theorem 4.2], to conclude that

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_t}{u} \leq \frac{c_1}{t}
\tag{4.12}$$

for some $c_1 > 0$.

With the gradient estimate (4.12), we can prove the following Harnack inequality for u .

Theorem 4.2. *For any $x, y \in M^n$ and $0 < t_1 < t_2$,*

$$\frac{u(y, t_2)}{u(x, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-c_1/2} \exp\left(-\frac{d_{g(t_1)}(x, y)^2}{4(t_2 - t_1)}(1 + t_2 - t_1)^{2\epsilon}\right).$$

Proof. Suppose $\gamma(t) : [t_1, t_2] \rightarrow M$ is a geodesic with respect to the metric $g(t_1)$ such that

$$|\dot{\gamma}(t)| = \frac{d_{g(t_1)}(x, y)}{t_2 - t_1}, \quad t_1 \leq t \leq t_2,$$

$$\gamma(t_1) = x, \quad \gamma(t_2) = y.$$

Then we have

$$\begin{aligned}
\log \frac{u(y, t_2)}{u(x, t_1)} &= \int_{t_1}^{t_2} \frac{d}{dt} (\log u(\gamma(t), t)) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial}{\partial t} \log u + \nabla \log u \cdot \frac{\partial \gamma}{\partial t} \right) dt \\
&\geq \int_{t_1}^{t_2} \left(\frac{|\nabla \log u|^2}{2} - \frac{c_1}{2t} + \nabla \log u \cdot \frac{\partial \gamma}{\partial t} \right) dt \quad \text{using (4.12)} \\
(4.13) \quad &\geq -\frac{c_1}{2} \log \left(\frac{t_2}{t_1} \right) - \frac{1}{2} \int_{t_1}^{t_2} \left| \frac{\partial \gamma}{\partial t} \right|_{g(t)}^2 dt.
\end{aligned}$$

Using the evolution equation of metric along Ricci flow and inequality (4.3),

$$\int_{t_1}^{t_2} \left| \frac{\partial \gamma}{\partial t} \right|_{g(t)}^2 dt \leq (1 + t_2 - t_1)^{2\epsilon} \int_{t_1}^{t_2} \left| \frac{\partial \gamma}{\partial t} \right|_{g(t_1)}^2 dt = (1 + t_2 - t_1)^{2\epsilon} \frac{d_{g(t_1)}(x, y)^2}{t_2 - t_1}$$

from the estimate (4.3).

Therefore (4.13) completes the proof. \square

Remark 4.3. We note that the proof of the above estimates does not depend on the order of decaying for the initial condition u_0 .

Theorem 4.4. *We have the following estimate. There exists $\delta > 0$ and $C > 0$ such that*

$$u(x, t) \leq \frac{C}{(1 + t)^{1+\delta}}.$$

Proof. We fix a constant $p \in (\frac{n}{2+\sigma}, \frac{n}{2})$, then from the decaying property (4.5) we have u^p is integrable and

$$\begin{aligned}
\frac{d}{dt} \left(\int u^p dV \right) &= \int (p u^{p-1} u_t - R u^p) dV \leq \int p u^{p-1} \Delta u dV \\
(4.14) \quad &= - \int p(p-1) u^{p-2} |\nabla u|^2 dV \leq 0
\end{aligned}$$

since $p > \frac{n}{2+\sigma} \geq 1$ and the integration by parts is justified by the decaying property of u and $|\nabla u|$.

So from (4.14) there exists $c_2 > 0$ such that

$$(4.15) \quad \int u^p dV \leq c_2.$$

For any $x \in M^n$ and $t \geq 1$ by using Harnack inequality,

$$\begin{aligned}
c_2 &\geq \int u^p dV \geq \int_{B_{g(t)}(x, (1+t)^{(1/2-\epsilon)})} u^p dV_{g(2t)} \\
&\geq \text{Vol}_{g(2t)}(B_{g(t)}(x, (1+t)^{(1/2-\epsilon)})) u^p(x, t) 2^{-c_1} \exp(-(1+t)/4t) \\
(4.16) \quad &\geq c_3 \text{Vol}_{g(2t)}(B_{g(t)}(x, (1+t)^{(1/2-\epsilon)})) u^p(x, t)
\end{aligned}$$

for some constant $c_3 > 0$.

The evolution equation for the volume of any compact set $K \subset M^n$ is

$$\frac{d}{dt} \left(\int_K dV \right) = \int_K -R dV \geq \frac{-\epsilon}{1+t} \int_K dV.$$

So we have

$$(4.17) \quad \text{Vol}_{g(t)}(K) \geq (1+t)^{-\epsilon} \text{Vol}_{g(0)}(K).$$

On the other hand, by the same reason

$$(4.18) \quad d_{g(t)}(x, y) \leq (1+t)^\epsilon d_{g(0)}(x, y)$$

for any $x, y \in M^n$.

So from (4.16) (4.17) and (4.18) we have

$$\begin{aligned}
c_2 &\geq c_3 \text{Vol}_{g(2t)}(B_{g(t)}(x, (1+t)^{(1/2-\epsilon)})) u^p(x, t) \\
&\geq c_3 (1+2t)^{-\epsilon} \text{Vol}_{g(0)}(B_{g(t)}(x, (1+t)^{(1/2-\epsilon)})) u^p(x, t) \\
&\geq c_3 (1+2t)^{-\epsilon} \text{Vol}_{g(0)}(B_{g(0)}(x, (1+t)^{(1/2-2\epsilon)})) u^p(x, t) \\
(4.19) \quad &\geq c_4 (1+2t)^{-\epsilon} (1+t)^{(1/2-2\epsilon)n} u^p(x, t)
\end{aligned}$$

for some $c_4 > 0$ by the AE condition of $g(0)$.

Hence we have

$$(4.20) \quad u(x, t) \leq C(1+t)^{\frac{\epsilon-(1/2-2\epsilon)n}{p}}.$$

Then if ϵ is sufficiently small which depends on p and n , then $\frac{\epsilon-(1/2-2\epsilon)n}{p} < -1$ and we can choose $\delta = -1 - \frac{\epsilon-(1/2-2\epsilon)n}{p} > 0$.

On the other hand if $t \leq 1$ the conclusion is obvious since u is uniformly bounded on compact time interval. \square

With Theorem 4.4, we prove the following estimate for the curvature operator.

Theorem 4.5. $|Rm| \leq \frac{C_0}{(1+t)^{1+\delta_0}}$ for some constants $C_0, \delta_0 > 0$.

Proof. Under Ricci flow, we have the following lemma by direct computation.

Lemma 4.6. Let T be a time-dependent tensor on M and u is a positive solution of $\partial_t u = \Delta_t u$, then

$$(\partial_t - \Delta) \frac{|T|^2}{u^2} = \frac{2}{u} \nabla u \cdot \nabla \frac{|T|^2}{u^2} - 2 \frac{|u \nabla T - \nabla u T|^2}{u^4} + \frac{(\partial_t - \Delta) |T|^2}{u^2}.$$

Let $W = \frac{|Rm|^2}{u^2}$, then from the Lemma 4.6 we have

$$\begin{aligned} \partial_t W &= \Delta W + \frac{2}{u} \nabla u \cdot \nabla W - 2 \frac{|u \nabla Rm - \nabla u Rm|^2}{u^4} + P \\ (4.21) \quad &\leq \Delta W + \frac{2}{u} \nabla u \cdot \nabla W + P, \end{aligned}$$

where

$$P = \frac{8(B_{ijkl} + B_{ikjl})R_{ijkl}}{u^2} \quad \text{and} \quad B_{ijkl} = -R_{pijq}R_{qlkp}.$$

We have the following estimate for P .

$$(4.22) \quad P \leq \frac{16|Rm|^3}{u^2} \leq \frac{16\epsilon}{1+t} W$$

where the last inequality is from (4.3).

Since $\frac{2}{u} \nabla u$ is uniformly bounded on $M^n \times [0, T]$ for any $T > 0$, by Theorem 2.1 we conclude that

$$(4.23) \quad W = \frac{|Rm|^2}{u^2} \leq C(1+t)^{16\epsilon}$$

for some constant $C > 0$.

Therefore, from Theorem 4.4 we know that there exists $C_0 = \sqrt{C} > 0$ such that

$$(4.24) \quad |Rm| \leq C_0 u (1+t)^{8\epsilon} \leq \frac{C_0}{(1+t)^{1+\delta-8\epsilon}}$$

where we can take $\delta_0 = \delta - 8\epsilon > 0$ by choosing ϵ to be small enough. \square

Now from the proof of Theorem 4.4, we know that for any σ_0 slightly smaller than σ ,

$$u(x, t) \leq C t^{-1-\sigma_0/2}$$

Therefore, $|Rm| \leq C t^{-1-\sigma_0/2}$. In other words, we have shown δ_0 can be chosen to be any number less than $\sigma_0/2$.

We have the following version of Shi's estimate, see also [36],

Theorem 4.7. *For any $k = 0, 1, \dots$*

$$|\nabla^k Rm| \leq C_k t^{-1-\delta_0-k/2}.$$

Proof. From the Theorem 4.5 the conclusion is true for $k = 0$. We assume by induction that it holds for any $0 \leq l < k$.

For any fix $s > 0$, we let

$$F = (t-s)^k |\nabla^k Rm|^2 + C_1 (t-s)^{k-1} |\nabla^{k-1} Rm|^2 + \dots + C_k |Rm|^2.$$

From the evolution equation of $|\nabla^k \text{Rm}|^2$

$$\begin{aligned}
 \partial_t |\nabla^k \text{Rm}|^2 &= \Delta |\nabla^k \text{Rm}|^2 - 2|\nabla^{k+1} \text{Rm}|^2 + \sum_{l=0}^k \nabla^l \text{Rm} * \nabla^{k-l} \text{Rm} * \nabla^k \text{Rm} \\
 (4.25) \quad &\leq \Delta |\nabla^k \text{Rm}|^2 - 2|\nabla^{k+1} \text{Rm}|^2 + C \sum_{l=0}^k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}|
 \end{aligned}$$

we have by the induction,

$$(t-s)^k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| \leq C t^{-2-2\delta_0} (t-s)^{k/2} |\nabla^k \text{Rm}| \leq C t^{-2-2\delta_0} F^{1/2}$$

for $0 < l < k$ and

$$(t-s)^k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| = (t-s)^k |\text{Rm}| |\nabla^k \text{Rm}|^2 \leq C t^{-1-\delta_0} F$$

for $l = 0$ or $l = k$.

Therefore, we can find nonnegative constants C_1, C_2, \dots, C_k , such that F satisfies the following equation

$$(4.26) \quad \partial_t F \leq \Delta F + C t^{-2-2\delta_0} (F^{1/2} + t^{1+\delta_0} F).$$

We consider the ODE

$$\begin{aligned}
 \frac{d\phi}{dt} &= C t^{-2-2\delta_0} (\phi^{1/2} + t^{1+\delta_0} \phi), \\
 (4.27) \quad \phi(s) &= C s^{-2-2\delta_0}
 \end{aligned}$$

such that $F(s) \leq \phi(s)$ since $F(s) = C_k |\text{Rm}|^2 \leq C_k s^{-2-2\delta_0}$.

Since $\phi(t)$ is increasing, $\phi(t) \geq C t^{-2-2\delta_0}$ for $t \geq s$ and hence

$$\frac{d\phi}{dt} \leq C t^{-1-\delta_0} \phi.$$

Then it is easy to show $\phi(t) \leq C s^{-2-2\delta_0}$ for $t \geq s$.

Now from Theorem 2.1, we conclude that

$$F(2s) \leq C s^{-2-2\delta_0}$$

since $F(s) = C_k |\text{Rm}|^2 \leq C_k s^{-2-2\delta_0}$.

In other words,

$$s^k |\nabla^k \text{Rm}|^2(2s) \leq C_k s^{-2-2\delta_0}.$$

Since s is an arbitrary positive number, we have

$$|\nabla^k \text{Rm}|(t) \leq C t^{-1-\delta_0-k/2}$$

which completes the induction process. \square

Thus there exist a metric g_∞ such that $g(t)$ converges to g_∞ smoothly as $t \rightarrow \infty$. Moreover, argue as before

$$\mu(g_\infty, \tau) \geq \limsup_{t \rightarrow \infty} \mu(g(t), \tau) \geq \limsup_{t \rightarrow \infty} \mu(g(0), \tau + t) = 0$$

for any $\tau > 0$.

Then from Theorem 3.3 we can show $(M^n, g_\infty) = (\mathbb{R}^n, g_E)$. In particular, M^n is diffeomorphic to \mathbb{R}^n .

5. PROOF OF THEOREM 1.2

In this section, we prove our first main theorem.

We first recall the definition of weighted function space, see for example [24]. Let (M, g) be an AE manifold with the AE end E , the weighted space $C_\beta^k(E)$ consists of C^k functions u for which the norm

$$\|u\|_{C_\beta^k} = \sum_{i=0}^k \sup_M r^{-\beta+i} |\nabla^i u|$$

is finite. The weighted Hölder space $C_\beta^{k,\alpha}(E)$ is defined for $0 < \alpha < 1$ as the set of $u \in C_\beta^k(E)$ for which the norm

$$\|u\|_{C_\beta^{k,\alpha}} = \|u\|_{C_\beta^k} + \sup_{x,y} (\min\{r(x), r(y)\})^{-\beta+k+\alpha} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x - y|^\alpha}$$

is finite.

Then we have the following convergence result in the weighted space.

Theorem 5.1. *For any $\sigma' \in (\frac{n-2}{2}, \sigma)$, we have $g_{ij}(t)$ converges to $g_{ij}(\infty)$ in $C_{-\sigma'}^\infty$ as $t \rightarrow \infty$. In particular, $(g_{ij}(\infty), E)$ is an AE coordinate system on M^n .*

Proof. We first prove a lemma

Lemma 5.2.

$$|\nabla^k Rm| \leq C_k t^{-1-\delta_k} r^{-k-\sigma'}$$

for some $C_k, \delta_k > 0$ and all $(x, t) \in M \times [0, \infty)$.

Proof of the lemma: We choose σ_1, σ_0 such that $\sigma' < \sigma_1 < \sigma_0 < \sigma$.

We consider a domain $D_k = \{(x, t) \in M \times [0, \infty) \mid r(x) \geq t^{a_k}\}$ in the spacetime where $a_k > 1/2$ to be determined later.

For $(x, t) \in D_k^c$, from Shi's estimates, we have

$$(5.1) \quad |\nabla^k Rm| \leq C_k t^{-1-\sigma_0/2-k/2} \leq C_k t^{-1-\delta_k} r^{-\sigma'-k}$$

for some $\delta_k > 0$ when a_k is sufficiently close to $1/2$.

For $(x, t) \in D_k$, we have the following estimate.

Claim: $|\nabla^k Rm|^2 \leq C r^{-4-2\sigma_1-2k}$ on D_k .

Proof of the claim: Let $h_k = r^{4+2\sigma_1+2k}$ and $w_k = h_k |\nabla^k Rm|^2$, like before we have

$$(5.2) \quad (\partial_t - \Delta)w_k \leq B_k w_k - 2\nabla \log h_k \nabla w_k + C \sum_{l=0}^k h_k |\nabla^l Rm| |\nabla^{k-l} Rm| |\nabla^k Rm|$$

where $B_k = \frac{2|\nabla h_k|^2 - h_k \Delta h_k}{h_k^2}$ is uniformly bounded by $r^{-2} \leq t^{-2a_k}$.

For $k = 0$, we have

$$(\partial_t - \Delta)w_0 \leq -2\nabla \log h_0 \nabla w_0 + C t^{-1-\delta'_0} w_0$$

for some $\delta'_0 = \min\{2a_0 - 1, \delta_0/2\} > 0$.

Moreover, on ∂D_0 we have

$$(5.3) \quad |\text{Rm}| \leq Ct^{-1-\sigma_0/2} = Cr^{-(1+\sigma_0/2)/a_0} \leq Cr^{-2-\sigma_1}$$

for a_0 sufficiently close to $1/2$.

Therefore, the claim holds for $k = 0$ from maximum principle.

Now we assume that the claim holds for all $0 \leq l < k$, then by induction on D_k we have

$$h_k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| = h_k |\text{Rm}| |\nabla^k \text{Rm}|^2 \leq t^{-1-\delta_0/2} w_k$$

for $l = 0$ or $l = k$ and

$$h_k |\nabla^l \text{Rm}| |\nabla^{k-l} \text{Rm}| |\nabla^k \text{Rm}| \leq Cr^k |\nabla^k \text{Rm}| = Cr^{-\sigma_1-2} w_k^{1/2} \leq Ct^{-a_k \sigma_1 - 2a_k} w_k^{1/2}$$

for $0 < l < k$.

Therefore from (5.2) we have

$$(\partial_t - \Delta) w_k \leq -2 \nabla \log h_k \nabla w_k + Ct^{-1-\delta'_k} (w_k + w_k^{1/2})$$

for some $\delta'_k > 0$.

On the other hand, on ∂D_k we have by Shi's estimate

$$(5.4) \quad |\nabla^k \text{Rm}| \leq C_k t^{-k/2} t^{-1-\sigma_0/2} = C_k r^{-(1+k/2+\sigma_0/2)/a_k} \leq C_k r^{-2-k-\sigma_1}$$

when a_k is chosen to be sufficiently close to $1/2$.

So from maximum principle, we conclude that w_k is uniformly bounded on D_k and the claim holds for k as well.

Therefore, on D_k we have

$$|\nabla^k \text{Rm}| \leq C_k r^{-2-k-\sigma_1} \leq C_k t^{-1-\delta_k} r^{-k-\sigma'}$$

for some $\delta_k > 0$ and a_k close to $1/2$.

Thus the proof of lemma is complete.

With the same argument in Theorem 2.2, we conclude that $g_{ij}(t)$ converges to $g_{ij}(\infty)$ in $C_{-\sigma'}^\infty$, because the term $t^{-1-\delta_k}$ guarantees that $|\nabla^k \text{Rm}|$ is integrable with respect to time at infinity. In other words, $g_{ij}(\infty)$ is an AE coordinate system with a smaller order σ' for the Euclidean space. \square

Now we continue to prove Theorem 1.2. We choose a smooth function η such that $\eta = 0$ outside of the AE end E and $\eta = 1$ when r is large.

Let $\chi(t) = (\partial_i g_{ij}(t) - \partial_j g_{ii}(t)) \partial_j$ be a vector field on the AE end, by the definition of mass,

$$\begin{aligned} m(g(t)) &= \lim_{r \rightarrow \infty} \int_{S^r} \chi(t) \lrcorner dx \\ &= \lim_{r \rightarrow \infty} \int_{S^r} \eta \chi(t) \lrcorner dx \\ (5.5) \quad &= \int \eta \text{div}(\chi(t)) + \langle \chi(t), \nabla \eta \rangle dx. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 R &= g^{jk}(\partial_i \Gamma_{jk}^i - \partial_k \Gamma_{ij}^i + \Gamma_{il}^i \Gamma_{jk}^l - \Gamma_{kl}^i \Gamma_{ij}^l) \\
 &= \partial_j(\partial_i g_{ij} - \partial_j g_{ii}) + O(r^{-2\sigma'-2}).
 \end{aligned}
 \tag{5.6}$$

So

$$\begin{aligned}
 |\operatorname{div} \chi(t) - \operatorname{div} \chi(\infty) - R(t)| &= |\operatorname{div} \chi(t) - \operatorname{div} \chi(\infty) - (R(t) - R(\infty))| \\
 &\leq C \|g(t) - g(\infty)\|_{C_{-\sigma'}^2} r^{-2\sigma'-2}.
 \end{aligned}
 \tag{5.7}$$

From (5.5) and (5.7) we have

$$\begin{aligned}
 m(g(0)) &= \lim_{t \rightarrow \infty} m(g(t)) = \lim_{t \rightarrow \infty} m(g(t)) - m(g(\infty)) \\
 &= \lim_{t \rightarrow \infty} \int \eta(\operatorname{div} \chi(t) - \operatorname{div} \chi(\infty)) + \langle \chi(t) - \chi(\infty), \nabla \eta \rangle \, dx \\
 &\geq \lim_{t \rightarrow \infty} \int \eta R(t) - C \eta \|g(t) - g(\infty)\|_{C_{-\sigma'}^2} r^{-2\sigma'-2} + \langle \chi(t) - \chi(\infty), \nabla \eta \rangle \, dx.
 \end{aligned}
 \tag{5.8}$$

Now since $\sigma' > \frac{n-2}{2}$, $\eta r^{-2\sigma'-2}$ is integrable. In addition, $\chi(t) - \chi(\infty)$ converges to 0 on the support of $\nabla \eta$ and $\|g(t) - g(\infty)\|_{C_{-\sigma'}^2}$ tends to 0, so we have from (5.8),

$$m(g(0)) \geq \lim_{t \rightarrow \infty} \int \eta R(t) \, dx \geq 0.
 \tag{5.9}$$

Remark 5.3. From the above proof, we can show

$$m(g(0)) = \lim_{t \rightarrow \infty} \int R(t) \, dV_t
 \tag{5.10}$$

since $g(t)$ converges to g_E uniformly on compact set.

If the equality holds, we have by (5.10) $\lim_{t \rightarrow \infty} \int R(t) \, dV_t = 0$.

On the other hand

$$\begin{aligned}
 \frac{d}{dt} \left(\int R \, dV \right) &= \int \Delta R + 2|Rc|^2 - R^2 \, dV \\
 &= \int 2|Rc|^2 - R^2 \, dV \\
 &\geq -\frac{n-2}{n} \int R^2 \, dV \quad (\text{from } |Rc|^2 \geq \frac{R^2}{n}) \\
 &\geq -\frac{C}{(1+t)^{1+\delta}} \int R \, dV
 \end{aligned}
 \tag{5.11}$$

where the second inequality holds since $\lim_{r \rightarrow \infty} \int_{S_r} |\nabla R(t)| \, dS = 0$ and hence $\int \Delta R \, dV = 0$ and last inequality follows from Theorem 4.5.

Then do the integration on both sides, $\lim_{t \rightarrow \infty} \int R(t) \, dV_t$ cannot be 0 unless $R(t) \equiv 0$, which is a contradiction by our original assumptions. In other words, the only possibility for $m(g(0)) = 0$ is when $(M^n, g) = (\mathbb{R}^n, g_E)$.

Thus, we have completed the proof of Theorem 1.2.

6. RICCI FLOW WITH SURGERY ON AE MANIFOLD

In this section, we define the Ricci flow with surgery on an AE manifold, most definitions and notations are from [32] [27] [7] and [22] with slight modifications. We assume from now on M is an orientable Riemannian AE 3-manifold with $R > 0$ unless otherwise specified.

First of all we fix a surgery model, see [32, Section 2] and [27, Chapter 12],

Definition 6.1. (surgery model) Consider $M_{\text{stan}} = \mathbb{R}^3$ with its natural $SO(3)$ -action, then there is a complete metric g_{stan} on M_{stan} such that

- (1) g_{stan} is $SO(3)$ -invariant.
- (2) g_{stan} has nonnegative sectional curvature.
- (3) There is a compact ball $B \subset M_{\text{stan}}$ so that the restriction of the metric g_{stan} to the complement of this ball is isometric to the product $(S^2, h) \times (\mathbb{R}^+, ds^2)$ where h is the round metric of scalar curvature 1 on S^2 .
- (4) There is a standard Ricci flow $(M_{\text{stan}}, g_{\text{stan}}(t)), 0 \leq t < 1$ such that 1 is the singular time.

For an AE manifold M^3 , under Ricci flow, we either have long time existence or the metric goes singular at some finite time. In the latter case, we modify the resulting limit by surgery, which cuts off high curvature parts and add standard capped tubes, so as to produce a new manifold with an AE end which serves a new initial condition for Ricci flow. Now we clarify the process of surgery at the first singular time for example.

Let $(M, g(t)), 0 \leq t < T$ be the Ricci flow solution where T is the first singular time. Let $\Omega \subset M$ be a subset defined by

$$\Omega = \{x \in M \mid \limsup_{t \rightarrow T} R_g(x, t) < \infty\}.$$

Then we have the following properties:

Theorem 6.2. (1) As $t \rightarrow T$ the metric $g(t)|_{\Omega}$ limit to $g(T)$ uniformly in the C^∞ -topology on every compact sets of Ω .

- (2) Every end of a connected component of Ω is contained in a strong ϵ -tube.
- (3) There exists $r > 0$ such that any $x \in \Omega \times \{T\}$ with $R(x) \geq r^{-2}$ has a strong (C, ϵ) -canonical neighborhood in $\widehat{M} = M \times [0, T) \cup_{\Omega \times [0, T)} (\Omega \times [0, T])$.
- (4) There exists a compact set $K \subset M$ such that $|Rm|$ is bounded on $K^c \times [0, T)$. In particular, $K^c \subset \Omega$.
- (5) The scalar curvature $R(g(T))$ is a proper function from $\Omega \rightarrow (0, \infty)$.

Proof. The proof of (1) – (3) can be found in [27, Theorem 11.9]. (4) is proved by pseudolocality, see [14, Theorem 1.1]. To prove (5), we need the following lemma.

Lemma 6.3. There exists a compact set K such that $g(T)$ is an AE coordinate system on $K^c = M - K$.

Proof of the lemma: From [14, Theorem 1.1], there exist a compact set K and $S > 0$ such that $|Rm(x, t)| \leq S$ on $K^c \times [0, T)$. Enlarge K if necessary, we can

assume $g_{ij}(0)$ is an AE coordinate system on K^c and ∂K is smooth. Then we can use the same argument in Theorem 2.2 on the parabolic cylinder $K^c \times [0, T)$ to conclude that $g(T)$ is an AE coordinate system on K^c .

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in Ω such that $0 < c \leq R(x_n, T) \leq C$ for some constants $0 < c < C$. Since by the Lemma 6.3, $g(T)$ has $r^{-2-\sigma}$ decaying at infinity, all x_n are contained in a compact set of M . Then we assume, by taking a subsequence if necessary, x_n converges to a point x_∞ in M . If x_∞ is not in Ω , by Lemma 7.2 in the next section, we have $R(x_n, T)$ goes to infinity which is a contradiction.

Thus, the proof of Theorem 6.2 is complete. \square

Remark 6.4. We call the end K^c in Lemma 6.3 AE end of Ω .

We fix $0 < \rho < r$ where r is the constant from Theorem 6.2(3) and define $\Omega_\rho \subset \Omega$ be the closed subset of all $x \in \Omega$ for which $R(x, T) \leq \rho^{-2}$. For a component Ω_1 of Ω which contains no point of Ω_ρ , by the canonical neighborhood theorem, one of the following holds, see [27, Lemma 11.28]:

- (1) Ω_1 is a strong double ϵ -horn and is diffeomorphic to $S^2 \times \mathbb{R}$.
- (2) Ω_1 is a C -capped ϵ -horn and is diffeomorphic to \mathbb{R}^3 .
- (3) Ω_1 is a compact component and is diffeomorphic to S^3/Γ or $S^1 \times S^2$.

Those are all possibilities as $S^1 \times_{\mathbb{Z}_2} S^2$ and punctured $\mathbb{R}P^3$ are excluded since M is orientable.

Let $\Omega^0(\rho)$ be the union of all components of Ω containing points of Ω_ρ , then $\Omega^0(\rho)$ has finitely many components and is a union of the AE end and finitely many strong ϵ -horns each of which is disjoint from Ω_ρ . The finiteness of horns can be derived from the properness of $R(T) \rightarrow (0, \infty)$ and the rest arguments can be found in [27, Lemma 11.30].

Next, we have the following lemma which asserts the existence of a strong δ -necks on which we will do surgeries.

Lemma 6.5. [7, Theorem 5.1] *For any $\delta > 0$, there exist $h \in (0, \delta\rho)$ and a constant $D = D(\delta, \rho)$ such that the following holds: Let $x, y, z \in \Omega$ such that $R(x, t) \leq \rho^{-2}$, $R(y, t) = h^{-2}$ and $R(z, t) \geq Dh^{-2}$. Assume there is a curve γ connecting x to z via y . Then (y, t) is center of a strong δ -neck.*

Now for the surgery parameters $r, \delta < 1$ we set $\rho = r\delta$, then the scale $h = h(\rho, r) = h(\delta, r)$ and $D = D(\rho, r) = D(\delta, r)$ are determined. Moreover, we require that

$$(6.1) \quad \lim_{\delta \rightarrow 0} \frac{D(\delta, r)h(\delta, r)}{\rho^6} = 0$$

since the proof of lemma 6.5 argues by contradiction by choosing two independent sequences $h_i \rightarrow 0$ and $D_i \rightarrow +\infty$.

We say (M_+, g_+) is obtained from $(\Omega, g(T))$ by (r, δ) -surgery at time T if

- (1) M_+ is obtained from Ω by removing components disjoint from Ω_ρ and cutting along a locally finite collection of disjoint 2-spheres, capping off 3-balls.
- (2) All $x \in M_+ \setminus M(T)$ are contained in a surgery cap and the cutting and capping are done on a strong δ -neck centered at a point y with $R(t, T) = h^{-2}$.

(3) (M_+, g_+) is pinched toward positive curvature.

Now we show (r, δ) -surgery must exist, see [7, Lemma 7.6].

By Zorn's lemma, on Ω there exists a maximal collection $\{N_i\}$ of pairwise disjoint δ -necks centered at y_i with $R(y_i, T) = h^{-2}$. Then from lemma (6.5), every components of $\Omega \setminus \cup_i N_i$ has the scalar curvature either less than Dh^{-2} or greater than ρ^{-2} . Then we remove all the components of the second kind and do surgeries on those δ -necks N_i .

Now we let M_+ be the resulting manifold and $R(g_+) \in (0, Dh^{-2}]$. From the construction we know that each component of M_+ contains at least one point p at which $R(p, T) \leq \rho^{-2}$, hence there are at most finitely many components by the properness of R . Moreover one of the component M_+^0 containing the AE end of M_+ is an AE manifold with the same order σ as M . In addition, the mass of $(M_+^0, g(T))$ is well defined and is equal to that of M , by the same argument in [16].

In general, we can construct by induction the weakly decreasing parameter functions $r, \delta, \kappa : [0, \infty) \rightarrow (0, \infty)$ to regulate the surgery process. The following existence theorem is proved in [27, Theorem 15.9], see also [7, Theorem 1.2].

Theorem 6.6. *There exists a Ricci flow with surgery $(\mathcal{M}, g_{\mathcal{M}})$ on $[0, \infty)$ with the initial condition (M, g) and decreasing functions $\delta(t), r(t), \kappa(t) : [0, \infty) \rightarrow \mathbb{R}^+$ such that the following holds,*

- (1) $(\mathcal{M}, g_{\mathcal{M}})$ has curvature pinched toward positive;
- (2) the flow satisfies the strong (C, ϵ) -canonical neighborhood theorem with parameter $r(t)$ on $[0, \infty)$;
- (3) the flow is $k(t)$ -noncollapsed on $[0, \infty)$ on scales $\leq \epsilon$ and
- (4) for any singular time t the surgery is performed with control $\delta(t)$ at scale $h(t) = h(\rho(t), \delta(t)) = h(r(t)\delta(t), \delta(t))$.

Next we show that surgery times do not accumulate.

Theorem 6.7. *Let (\mathcal{M}, G) be a Ricci flow with surgery on $[0, \infty)$ with the initial condition (M, g) with parameter functions $\delta(t), r(t), \kappa(t)$, we show that on each compact interval I of $[0, \infty)$, we have at most finitely many surgeries.*

Proof. Since all the parameter functions are decreasing, we can choose uniform parameters δ, r and κ on I . Therefore functions h and D are uniformly determined as well. At each singular time t , by our construction $R(x, t) \leq Dh^{-2}$. Since curvature is pinched toward positive curvature, we can assume $|\text{Rm}| \leq CDh^{-2}$. Now from the evolution equation of $|\text{Rm}|^2$

$$\partial_t |\text{Rm}|^2 \leq \Delta |\text{Rm}|^2 + 16 |\text{Rm}|^3$$

the regular Ricci flow exists at least for time $\frac{h^2}{16CD}$ from t . Since all constants are uniformly chosen, there are at most finitely many surgeries performed on I . \square

Remark 6.8. Theorem 6.7 holds for all Ricci flows with surgery with normalized initial condition, which is satisfied after a scaling, if necessary, for our original manifold M .

From the construction of Ricci flow with surgery, each time slice $(\mathcal{M}(t), g(t))$ consists of an AE manifold and a finite number of compact components. Moreover, we can recover the topology of $\mathcal{M}(0) = M$ by performing connected sum operations among $\mathcal{M}(t)$ and finitely many S^3/Γ and $S^1 \times S^2$ for any $t > 0$.

7. PROOF OF THEOREM 1.3

We first have the following definition.

Definition 7.1. For a Ricci flow with surgery \mathcal{M} , a connected open subset $\mathcal{X} \subset \mathcal{M}$ is called a path of components if for every time t , the intersection $\mathcal{X}(t)$ of \mathcal{X} with each time-slice $\mathcal{M}(t)$ is a connected component of $\mathcal{M}(t)$.

We set \mathcal{M}_0 to be the path of components of \mathcal{M} such that $\mathcal{M}_0(t)$ is an AE manifold for any $t \geq 0$.

Next we quote a local regularity lemma.

Lemma 7.2. [23, Lemma 3.1] *Let \mathcal{M} be a Ricci flow with surgery, with normalized initial condition. Given $T > \frac{1}{100}$, there are numbers $\mu = \mu(T) \in (0, 1)$, $\sigma = \sigma(T) \in (0, 1)$, $i_0 = i_0(T) > 0$ and $A_k = A_k(T) < \infty$, $k \geq 0$, with the following property. If $t \in (\frac{1}{100}, T]$ and $|R(x, t)| < \mu\rho(0)^{-2} - r(T)^{-2}$, put $Q = |R(x, t)| + r(t)^{-2}$. Then*

- (1) *The forward/backward parabolic ball $P_{\pm}(x, t, \sigma Q^{-\frac{1}{2}})$ is unscathed, that is, with no intersection with the surgery cap.*
- (2) *$|Rm| \leq A_0 Q$, $\text{inj} \geq i_0 Q^{-\frac{1}{2}}$ and $|\nabla^k Rm| \leq A_k Q^{1+\frac{k}{2}}$ on the union $P_+(x, t, \sigma Q^{-\frac{1}{2}}) \cup P_-(x, t, \sigma Q^{-\frac{1}{2}})$ of the forward and backward parabolic balls.*

Now we take a sequence of $\{\mathcal{M}^i\}$ of Ricci flows with surgery, where we let $\delta_i(0) \rightarrow 0$, hence ρ_i and h_i also go to 0. We first prove a stability result, which shows that on the finite time interval, all surgeries are done in a compact set.

Theorem 7.3. *Let $\{\mathcal{M}^i\}$ be a sequence of Ricci flows with surgery with $\mathcal{M}^i(0) = M$ and $\lim_{i \rightarrow \infty} \delta_i(0) = 0$. For any $S > 0, T > 0$, there exists a compact set $K \subset M$ such that for sufficiently large i , the cylinder $K^c \times [0, T]$ exists in \mathcal{M}^i and $|Rm_i| \leq S$.*

Proof. We prove it by contradiction.

Assume there is a sequence $(x_j)_{j \in \mathbb{N}}$ on M with $d_g(x_j, \star) = 2r_i$ where \star is a fixed point on M and $r_i \rightarrow \infty$ such that $|Rm_j|(x_j, t_j) > S$ for some $t_j \in [0, T]$.

By the AE condition, balls $(B_g(x_j, r_j), g, x_j)$ converges smoothly to $(\mathbb{R}^n, g_E, 0)$. Then there exists a $\theta > 0$ sufficiently small such that $B_g(x_j, r_j) \times [0, \theta]$ exists in \mathcal{M}^j and for any $A > 0$, restriction of G_j on $B_g(x_j, A) \times [0, \theta]$ converges smoothly to the Euclidean metric on $B_{g_E}(0, A) \times [0, \theta]$.

Therefore for any $A > 0$, we assume $|Rm| \leq S/2$ on $B_g(x_j, A) \times [0, \theta]$ for j sufficiently large. From Lemma 7.2, there exists $Q, \sigma, A_k, \theta' = \sigma Q^{-\frac{1}{2}}$, all of which depend on $S, T, r, \kappa, (M, g)$, such that the forward parabolic ball $P_+(x_j, \theta, \theta')$ and the backward parabolic ball $P_-(x_j, \theta, \theta')$ are unscathed and $|\nabla^k Rm| \leq A_k Q^{1+\frac{k}{2}}$ with $\text{inj} \geq i_0 Q^{-\frac{1}{2}}$ on $B_g(x_j, A) \times [\theta - \theta', \theta + \theta']$ for j sufficiently large. By taking a diagonal subsequence, we have $B_g(x_j, r_j) \times [0, \theta + \theta']$ converges smoothly to the Euclidean metric on $\mathbb{R}^n \times [0, \theta + \theta']$.

Now we can continue this process, since θ' does not depend on δ^j , to conclude that $B_g(x_j, r_i) \times [0, T]$ converges smoothly to the Euclidean metric on $\mathbb{R}^n \times [0, T]$ and $|\text{Rm}_j| \leq S/2$ on $B_g(x_j, 1) \times [0, T]$. This is a contradiction. \square

By the evolution equation for the volume $\partial_t dV_t = -RdV_t \leq 0$ and the fact that the surgery process is volume-decreasing, it is true that for any compact set $A \subset M$, $\text{Vol}_t(A)$ is decreasing.

From Theorem 3.3, we can find a constant $\epsilon_0 > 0$ such that $\mu_{S^2 \times \mathbb{R}}(g_c, 1) \leq -2\epsilon_0$, where g_c is the standard metric on the cylinder with scalar curvature $R = 1$. Therefore, we choose the parameter ϵ for the surgery as follows, for any ϵ -neck with metric g and center p , we have $\mu_{S^2 \times (-\epsilon^{-1}, \epsilon^{-1})}(R(p)g, 1) \leq -\epsilon_0$.

From Theorem 3.4, there exists a constant $T > 0$ such that

$$(7.1) \quad \mu_M(g, \tau) \geq -\epsilon_0/2$$

for any $\tau \geq T$.

From Theorem 7.3, there exists a compact set $K \subset M$ such that $|\text{Rm}_i| \leq 1$ on $(M \setminus K) \times [0, T]$ and we can find a common AE coordinate system for all $g_i(T)$. Moreover from maximum principle it is easy to show that $\mathcal{M}_0^i(T) \setminus (M - K)$ have uniform positive lower bound of scalar curvature. Hence, from Theorem 3.7 there exists $T' > T$ such that

$$(7.2) \quad \mu_M(g_i(T), \tau) \geq -\epsilon_0/2$$

for any $\tau \geq T' - T$ and i .

Now we take a sequence of Ricci flow with surgery $\{\mathcal{M}^i\}$ with initial condition (M, g) subject to a uniform $r(t) > 0$ and surgery parameter $\delta_i(0) \rightarrow 0$. Since all $r(t)$ and $\delta_i(t)$ are decreasing, we can choose $r > 0$, $\delta_i \rightarrow 0$ as constant parameters on the time interval $[0, T']$.

With all those preparations, Theorem 1.3 follows immediately from Theorem 1.2 and the following theorem.

Theorem 7.4. *There are finitely many surgeries for \mathcal{M}_0^i for i sufficiently large.*

Proof. Suppose the conclusion is false. Then we can assume for all i , \mathcal{M}_0^i has infinitely many surgeries. In particular, we denote the first surgery time past T by $T_{k_i}^i$ for \mathcal{M}_0^i and all previous surgery times by $\{T_1^i, T_2^i, \dots, T_{k_i-1}^i\}$. We also set $(\sigma_j^i)^2 = T_{k_i}^i - T_{k_i-j}^i$ for $1 \leq j \leq k_i$ and $T_0^i = 0$.

If $T_{k_i}^i \geq T'$, as $T_{k_i}^i$ is a singular time, we can find a sequence of points $\{p_j^i = (x_j^i, t_j^i)\}_{j \in \mathbb{N}}$ in \mathcal{M}_0^i such that $t_j^i \rightarrow T_{k_i}^i$ and if $Q_j^i = R(x_j^i, t_j^i)$, $(\mathcal{M}_0^i(t_j^i), Q_j^i g(t_j^i), x_j^i)$

converges smoothly as $j \rightarrow \infty$ to a standard cylinder $(S^2 \times \mathbb{R}, g_c)$. Then we have

$$\begin{aligned}
-2\epsilon_0 &\geq \mu_{S^2 \times \mathbb{R}}(g_c, 1) \\
&\geq \lim_{j \rightarrow \infty} \mu(Q_j^i g_i(t_j^i), 1) \\
&= \lim_{j \rightarrow \infty} \mu(g_i(t_j^i), 1/Q_j^i) \\
&\geq \lim_{j \rightarrow \infty} \mu(g_i(T), 1/Q_j^i + t_j^i - T) \\
(7.3) \quad &= \mu(g_i(T), T_{k_i}^i - T)
\end{aligned}$$

which contradicts (7.2) since $T_{k_i}^i - T \geq T' - T$.

Therefore, we can assume all $T_{k_i}^i \leq T'$.

By the same point-picking method as above, we have

$$(7.4) \quad \mu(g(T_{k_i-1}^i), (\sigma_1^i)^2) = \mu(g(T_{k_i-1}^i), T_{k_i}^i - T_{k_i-1}^i) \leq -2\epsilon_0$$

To estimate μ past surgery time $T_{k_i-1}^i$, we have the following two cases.

Case 1: $\sigma_1^i \leq \rho_i$.

In this case, since $T_{k_i-1}^i$ is a singular time, we can find a point p with $R(p) = (\sigma_1^i)^{-2}$, which is the center of an ϵ -neck. By our choice of ϵ , we have $\mu(g(T_{k_i-1}^i), (\sigma_1^i)^2) \leq -\epsilon_0$, where $g(T_{k_i-1}^i)$ is the metric on the manifold before surgery at $T_{k_i-1}^i$.

Case 2: $\sigma_1^i \geq \rho_i$.

In this case, we have the following estimate for μ , which is from [42, (2.21)].

$$(7.5) \quad \mu(g(T_{k_i-1}^i), (\sigma_1^i)^2) \leq \mu(g(T_{k_i-1}^i), (\sigma_1^i)^2) + c \frac{\int_U u^2 d\tilde{g}}{1 - \int_U u^2 d\tilde{g}}$$

where u is a minimizer of $\mu(g(T_{k_i-1}^i), (\sigma_1^i)^2)$, $\tilde{g} = (\sigma_1^i)^{-2} g(T_{k_i-1}^i)$ and U is the union of finitely many (denote the number by k) surgery caps

We first estimate k . On $\mathcal{M}_0^i(T_{k_i-1}^i)$, we can find k disjoint ϵ -tubes and each contains an ϵ -neck with center p and $R(p) = \rho_i^{-2}$. The total volume of all k tubes are at least $ck\rho_i^3$. Since all surgeries are done in a compact set K whose volume is decreasing along the flow, we have

$$(7.6) \quad k \leq C\rho_i^{-3}.$$

Since $\sigma_1^i \geq \rho_i$, we know from [42, Lemma 2.1] that

$$(7.7) \quad u^2 \leq C \frac{(\sigma_1^i)^3}{\rho_i^3}.$$

Combining (7.6) and (7.7), we have

$$(7.8) \quad \int_U u^2 d\tilde{g} \leq Ck \frac{(\sigma_1^i)^3}{\rho_i^3} \frac{h_i^3}{(\sigma_1^i)^3} \leq C \frac{h_i^3}{\rho_i^6}.$$

Since h_i is much smaller than ρ_i , we have from (7.5)

$$(7.9) \quad \mu(g(T_{k_i-1}^i), (\sigma_1^i)^2) \leq \mu(g(T_{k_i-1}^i), (\sigma_1^i)^2) + c \frac{\int_U u^2 d\tilde{g}}{1 - \int_U u^2 d\tilde{g}} \leq -2\epsilon_0 + C \frac{h_i^3}{\rho_i^6}.$$

Therefore in both cases, we have

$$\mu(g((T_{k_i-1}^{i-}), (\sigma_1^i)^2) \leq -\epsilon_0 + C \frac{h_i^3}{\rho_i^6}.$$

By iteration from 1 to k_i , we have

$$(7.10) \quad \mu(g(0), (\sigma_{k_i}^i)^2) \leq -\epsilon_0 + C k_i \frac{h_i^3}{\rho_i^6}.$$

We know that from Theorem 6.7, the gap of two consecutive surgeries is at least $CD_i^{-1}h_i^2$, then

$$(7.11) \quad k_i \leq CD_i T' h_i^{-2}.$$

Hence from (7.10), in the latter case,

$$(7.12) \quad \mu(g(0), (\sigma_{k_i}^i)^2) \leq -\epsilon_0 + CT' \frac{D_i h_i}{\rho_i^6}.$$

From our choice of parameters, i.e. (6.1), $\lim_{i \rightarrow \infty} \frac{D_i h_i}{\rho_i^6} = 0$, so for i sufficiently large, $CT' \frac{D_i h_i}{\rho_i^6} \leq \epsilon_0/3$.

Therefore, in both cases, we have $\mu(g(0), (\sigma_{k_i}^i)^2) \leq -2\epsilon_0/3$. But this contradicts (7.1) since $(\sigma_{k_i}^i)^2 \geq T$.

Thus, the proof of Theorem 7.4 is complete. \square

Proof of Theorem 1.3: If the Ricci flow has long time existence, the result follows from Theorem 1.2. Therefore we can assume that the Ricci flow has at least one surgery. From Theorem 7.4, if the surgery control function is chosen to be small enough, there are only finitely many surgeries and hence after the last surgery time the Ricci flow has long time existence. Since the mass is preserved along Ricci flow and surgery times, the mass is nonnegative by Theorem 1.2. If the equality holds, then again from Theorem 1.2, the metric after the last surgery is Euclidean, which is impossible.

Corollary 7.5. [35, Corollary 6] *Any orientable AE 3-manifold M with scalar curvature $R \geq 0$ has the following diffeomorphism type*

$$M \cong \mathbb{R}^3 \# S^3/\Gamma_1 \# \dots \# S^3/\Gamma_k \# (S^2 \times S^1) \# \dots \# (S^2 \times S^1)$$

where there are finitely many connected sums.

Proof. From Theorem 7.4, we have a Ricci flow with surgery \mathcal{M} such that there are only finitely many surgeries on \mathcal{M}_0 . After a large time T , the Ricci flow on $\mathcal{M}_0(T)$ has longtime existence, each of whose timeslice by Theorem 1.2 is diffeomorphic to \mathbb{R}^3 . Moreover, at time T , all other finitely many components of $\mathcal{M}(T)$ are compact manifolds with $R > 0$. Therefore they must extinct after finite time. Therefore we can recover the diffeomorphism type of M by performing connected sum of \mathbb{R}^3 with finitely many S^3/Γ and $S^2 \times S^1$. \square

Remark 7.6. Robert Haslhofer obtained the same result, see details in [35, Corollary 6], by using the min-max argument of Colding-Minicozzi [13].

A natural question is whether we have the same result if we only assume $g_{ij} - \delta_{ij} \in C^2_{-\sigma}$.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor, Professor Bing Wang. He brought this problem to my attention and steered me in the right direction. I'm also grateful to Professor Xiuxiong Chen and Professor Gábor Székelyhidi for their helpful discussions. Finally, I want to thank my parents and Yumeng Wang, for all their love and support.

APPENDIX A. GRADIENT RICCI SOLITON ON ALE MANIFOLD

In this section we prove some facts about Ricci gradient soliton on ALE manifold.

Definition A.1. A smooth Riemannian manifold (M^n, g) is called an asymptotically locally Euclidean (ALE) end of order $\sigma > 0$ if there exist a finite subgroup $\Gamma \subset O(n)$ acting freely on $\mathbb{R}^n \setminus B(0, R)$, a compact set $K \subset M^n$ and a C^∞ diffeomorphism $\Phi : M^n \setminus K \rightarrow (\mathbb{R}^n \setminus B(0, R))/\Gamma$ such that under this identification,

$$(A.1) \quad g_{ij} = \delta_{ij} + O(r^{-\sigma}),$$

$$(A.2) \quad \partial^{|k|} g_{ij} = O(r^{-\sigma-k}),$$

for any partial derivatives of order k as $r \rightarrow \infty$, where r is the Euclidean distance. A complete, noncompact manifold (M^n, g) is called ALE if M^n can be written as the disjoint union of a compact set and finitely many ALE ends [8] [40]. For an ALE end, if the group Γ in the definition is trivial, we call it a *trivial end* or AE end, otherwise we call it a *nontrivial end*.

Definition A.2. A metric g for a manifold M^n is called a gradient Ricci soliton if there is a smooth function $f : M^n \rightarrow \mathbb{R}$ such that

$$(A.3) \quad Ric + Hess(f) + \lambda g = 0.$$

It is called steady when $\lambda = 0$, shrinking when $\lambda = -1$ and expanding when $\lambda = 1$.

In [20] R. Hamilton proved the following identity for gradient steady Ricci solitons

$$(A.4) \quad R + |\nabla f|^2 = \Lambda$$

where Λ is a constant. Since on ALE manifold scalar curvature $R = O(r^{-2-\sigma})$, $|\nabla f|$ is bounded from (A.4). Therefore the metric g is a complete steady Ricci soliton, i.e., the vector field ∇f is complete. By the standard result in Ricci flow, for example in [12], there exists an eternal solution $g(t)$ ($-\infty < t < \infty$) of the Ricci flow with $g(0) = g$ such that $g(t) = \phi(t)^*g$ where $\phi(t)$ is the 1-parameter family of diffeomorphisms generated by ∇f .

Since the solution $g(t)$ is self-similar, its curvature operator $Rm(t)$ is uniformly bounded as $Rm(0)$ is bounded for ALE manifold. Moreover, $R \geq 0$ for every bounded ancient solution of Ricci flow, see [9]. By the strong maximum principle either $R > 0$ or M is Ricci-flat. In the first case, by rescaling we can assume

$$(A.5) \quad R + |\nabla f|^2 = 1.$$

If the steady gradient Ricci soliton is nontrivial, the manifold has to be one-ended, see [29].

Then we have

Theorem A.3. *If (M^n, g) is an ALE manifold such that g is a gradient steady Ricci soliton, then g is Ricci-flat.*

Proof. (Nontrivial end) If M^n is not Ricci-flat, we assume (A.5) holds.

We assume the end $E = (\mathbb{R}^n \setminus B(0, R))/\Gamma$ is a nontrivial end, that is, $|\Gamma| > 1$. By taking trace of (A.3), we have

$$(A.6) \quad R + \Delta f = 0$$

We consider the annulus $C_{a,A} = \{x \in E \mid a < d(x) = d(x, O) < A\}$ where O is a fixed point in M^n and a a fixed large number. On the geodesic sphere $S_A = \{x \in E \mid d(x) = A\}$ there is a point c such that $f(c) = \max_{x \in S_R} f(x)$ and hence $\nabla f(c)$ and $\nabla r(c)$ are parallel. From conditions on ALE end, $g \rightarrow g_E$, $r \rightarrow d$, $|\nabla f| \rightarrow 1$, and $\nabla r \rightarrow \nabla d$ uniformly when $d(x) \rightarrow \infty$. So if A is sufficiently large, we can assume

$$(A.7) \quad \langle \nabla f(c), \nabla d(c) \rangle \geq 1 - \epsilon$$

or

$$(A.8) \quad \langle \nabla f(c), \nabla d(c) \rangle \leq -(1 - \epsilon)$$

for a small number $\epsilon > 0$.

If (A.7) holds, for any point b on S_A we can find a minimal geodesic $\gamma(t)$, $0 \leq t \leq l$ such that $\gamma(0) = c$, $\gamma(l) = b$ and $|\dot{\gamma}(t)| = 1$. Moreover, we have

$$(A.9) \quad \begin{aligned} \frac{d\langle \nabla f(\gamma(t)), \nabla d(\gamma(t)) \rangle}{dt} &= \text{Hess}(f)(\dot{\gamma}, \nabla d) + \text{Hess}(d)(\nabla f, \dot{\gamma}) \\ &= -\text{Ric}(\dot{\gamma}, \nabla d) + \text{Hess}(d)(\nabla f, \dot{\gamma}) \end{aligned}$$

Since $|Rm| = O(r^{-2-\sigma})$ and by curvature comparison theorem, when A is large

$$(A.10) \quad -\epsilon g \leq \text{Ric} \leq \epsilon g$$

$$(A.11) \quad \frac{1-\epsilon}{A} g_A \leq \text{Hess}(d) \leq \frac{1+\epsilon}{A} g_R$$

when $x \in S_A$ and g_A is the metric on S_A induced by g .

So from (A.9) (A.10) (A.11), we have

$$(A.12) \quad \begin{aligned} \frac{d\langle \nabla f(\gamma(t)), \nabla d(\gamma(t)) \rangle}{dt} &\geq -\epsilon |\nabla d| + \frac{1-\epsilon}{A} g_A(\nabla f, \dot{\gamma}) \\ &\geq -\epsilon + \frac{1-\epsilon}{A} g_A(\langle \nabla f, \dot{\gamma} \rangle \dot{\gamma}, \dot{\gamma}) \\ &= -\epsilon + \frac{1-\epsilon}{A} \langle \nabla f, \dot{\gamma} \rangle \\ &= -\epsilon + \frac{1-\epsilon}{A} \sqrt{|\nabla f|^2 - \langle \nabla f, \nabla d \rangle^2} \\ &\geq -\epsilon + \frac{1-\epsilon}{A} \sqrt{1 - \langle \nabla f, \nabla d \rangle^2}. \end{aligned}$$

Compare (A.12) with ODE $\frac{dy}{dt} = \frac{1}{A} \sqrt{1-y^2}$ considering the initial value (A.7), we have when $A \rightarrow \infty$, $\epsilon \rightarrow 0$, there is an $\epsilon_0 > 0$ such that

$$(A.13) \quad \langle \nabla f(\gamma(t)), \nabla d(\gamma(t)) \rangle \geq (1 - \epsilon_0) \sin\left(\frac{\pi}{2} + \frac{t}{A}\right)$$

Since $|\Gamma| > 1$, if A is large, any point b on S_A can be connected to c by a minimal geodesic having the length at most $\frac{\pi A}{|\Gamma|}$. Therefore from (A.13) there exists a $c_0 > 0$ such that for large A

$$(A.14) \quad \int_{S_A} \langle \nabla f, \nabla d \rangle d\sigma \geq c_0 A^{n-1}$$

Similarly, if (A.8) holds at first, we have

$$(A.15) \quad \int_{S_A} \langle \nabla f, \nabla d \rangle d\sigma \leq -c_0 A^{n-1}$$

From the continuity, either (A.14) or (A.15) must hold for all large A .

If (A.14) holds, we have

$$(A.16) \quad \begin{aligned} 0 &> \int_{C_{a,A}} -R d\sigma = \int_{C_{a,A}} \Delta f d\sigma \\ &= \int_{S_A} \langle \nabla f, \nabla d \rangle d\sigma - \int_{S_a} \langle \nabla f, \nabla d \rangle d\sigma \\ &\geq c_0 A^{n-1} - C_1 \end{aligned}$$

for some constant $C_1 > 0$. But we have a contradiction if $A \rightarrow \infty$.

If (A.15) holds, we have

$$(A.17) \quad \begin{aligned} \int_{C_{a,A}} -R dV &= \int_{C_{a,A}} \Delta f d\sigma \\ &= \int_{S_A} \langle \nabla f, \nabla d \rangle d\sigma - \int_{S_a} \langle \nabla f, \nabla d \rangle d\sigma \\ &\leq -c_0 A^{n-1} + C_1. \end{aligned}$$

Since $R = O(d^{-2-\sigma})$,

$$(A.18) \quad \int_{C_{a,A}} -R dV \geq -C_2 \int_a^A s^{-2-\sigma} s^{n-1} ds \geq -C_3 - C_4 A^{n-2-\sigma}$$

for some constant $C_2, C_3, C_4 > 0$. So from (A.17) and (A.18), we also have a contradiction if $A \rightarrow \infty$.

(Trivial end): Assume the ALE end E of M^n is trivial. From Theorem 3.3, we can assume for all $\tau > 0$, $\mu(g, \tau) < 0$ since the Ricci flow solution of the steady soliton is eternal and M^n is not Ricci-flat.

For any $\bar{\tau} > 0$, by the monotonicity formula, $\mu(g(t), \bar{\tau} - t)$ is increasing for all $0 \leq t < \bar{\tau}$. Therefore

$$\mu(g(t), \bar{\tau} - t) = \mu(\phi(t)^* g, \bar{\tau} - t) = \mu(g, \bar{\tau} - t)$$

is increasing for all $0 \leq t < \bar{\tau}$. Since $\bar{\tau}$ can be any positive number, $\mu(g, \tau)$ is decreasing for all $\tau > 0$. So it contradicts Theorem 3.4. Thus, the proof of Theorem A.3 is complete. \square

For a complete Ricci shrinking soliton, we have

Theorem A.4. *If (M^n, g) is an ALE manifold such that g is a gradient shrinking Ricci soliton, then $(M^n, g) = (\mathbb{R}^n, g_E)$.*

The proof of A.4 follows immediately from the next theorem since by the ALE condition $|\text{Rm}| \leq Cr^{-2-\sigma}$.

Theorem A.5. *Let $(M^n, g(t)), t \in (-\infty, 0]$ be a non-flat κ -noncollapsed type-I ancient solution, that is, $|\text{Rm}|(x, t) \leq \frac{D}{1+|t|}$ for all $t \leq 0$. Then we have*

$$\limsup_{d_0(x, O) \rightarrow \infty} |\text{Rm}|(x, 0) d_0^2(x, O) > 0$$

for a fixed point O .

Proof. We assume the contrary. Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of points going to infinity such that $|\text{Rm}|(x_i, 0) \lambda_i \rightarrow 0$ where $\lambda_i = d_0^2(x_i, O)$. Then from [15, Theorem 4.1], $(M, \frac{1}{\lambda_i} g(\lambda_i t), x_i)$ converges smoothly to a nonflat shrinking soliton $(M_\infty, g_\infty(t), x_\infty)$ for $t < 0$.

By our assumption and the κ -noncollapsed condition, we conclude further that $(B_{\lambda_i^{-1}g(0)}(x_i, 1/2), \lambda_i^{-1}g(\lambda_i t), x_i)$ converges to $(B_{g_\infty(0)}(x_\infty, 1/2), g_\infty(t), x_\infty)$ for $t \leq 0$ from Cheeger-Gromov compactness theorem. Then we have $\text{Rm}_\infty(0) = 0$ on the metric ball $B_{g_\infty(0)}(x_\infty, 1/2)$. Since any shrinking soliton has nonnegative scalar curvature, by the strong maximum principle, $R_\infty(t) = 0$ for any $t < 0$ and hence Ricci-flat from $\partial_t R = \Delta R + 2|Rc|^2$. Then from $\text{Ric}_\infty + \text{Hess}(f_\infty) - \frac{g_\infty}{2} = 0$, we have $\text{Hess}(f_\infty) = \frac{g_\infty}{2}$. Therefore $(M_\infty, g_\infty) = (\mathbb{R}^n, g_E)$ by the same argument in Theorem 3.3.

Therefore we have a contradiction. \square

Remark A.6. It was proved in [10] that $\liminf_{d(x, O) \rightarrow \infty} R(x) d^2(x, O) > 0$ for any non-flat shrinking soliton.

There are nontrivial examples of expanding soliton on ALE manifold, see the constructions in [28].

REFERENCES

- [1] Robert A. Adams, *Sobolev Spaces*, Volume 140, Second Edition (Pure and Applied Mathematics), July 15, 2003.
- [2] Aubin, Thierry, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry 11 (1976), no. 4, 573-598.
- [3] Greg Anderson, Bennett Chow, *A pinching estimate for solutions of the linearized Ricci flow system on 3-manifolds*, arXiv:math/0211210.
- [4] R. Arnowitt, S. Deser, and C. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. 122, 997-1006 (1961).
- [5] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. 39, 661-693 (1986).
- [6] Richard H. Bamler, *Ricci flow with surgery*, diploma thesis, Ludwig-Maximilians-Universit at Munich (2007).
- [7] L. Bessieres, G. Besson, S. Maillot, *Ricci flow on open 3-manifolds and positive scalar curvature*, Geom. Topol. 15, 927-975 (2011).

- [8] S.Bando, A.Kasue, H.Nakajima, *On a construction of coordinates at infinity on manifold with fast curvature decay and maximal volume growth*, Invent. Math., 97(1989)313-349.
- [9] Chen, B.L., *Strong uniqueness of the Ricci flow*, J. Differential Geom.82(2009), 363382, MR 2520796, Zbl 1177.53036.
- [10] B. Chow, P. Lu, B Yang, *A lower bound for the scalar curvature of noncompact nonflat Ricci shrinkers*, C. R. Math. Acad. Sci. Paris 349(2011), 1265-1267.
- [11] Chow, Bennett; Chu, Sun-Chin; Glickenstein, David; Guenther, Christine; Isenberg,James; Ivey, Tom; Knopf, Dan; Lu, Peng; Luo, Feng; Ni, Lei, *The Ricci flow: techniques and applications. Part II. Geometric-analytic aspects* Mathematical Surveys and Monographs, 163. American Mathematical Society, Providence, RI, 2010.
- [12] Bennett Chow, Peng Lu, Lei Ni, *Hamilton's Ricci Flow*, Lecture in Contemporary Mathematics, 3, Science Press and Graduate Studies in Mathematics, 77, merican Mathematical Society, 2006.
- [13] T. Colding, W. Minicozzi, *Width and finite extinction time of Ricci flow*, Geom.Topol. 12, no. 5, 2537-2586 (2008).
- [14] Albert Chau, Luen-Fai Tam, Chengjie Yu, *Pseudolocality for the Ricci Flow and Applications*, Canad. J. Math. 63(2011), 55-85.
- [15] X. Cao, Q. S. Zhang, *The conjugate heat equation and ancient solutions of the Ricci flow*, arXiv:1006.0540v1 (2010).
- [16] X. Dai, L. Ma, *Mass under the Ricci flow*, Comm. Math. Phys. 274, no. 1, 65-80(2007).
- [17] F.B. Weissler, *Logarithmic Sobolev inequalities for the heat-diffusion semigroup*, Trans. Amer. Math. Soc. 237 (1978) 255-269.
- [18] Leonard Gross, *Logarithmic Sobolev Inequalities*, American Journal of Mathematics Vol. 97, No. 4 (Winter, 1975), pp. 1061-1083.
- [19] G. Huisken, T. Ilmanen, *The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality*, J. Diff. Geom., 59(3):353-437, 2001.
- [20] Richard S. Hamilton, *The Formation of Singularities in the Ricci Flow*, Surveys in Differential Geometry(Cambridge, MA, 1993), Vol.2, 7-136, International Press, Combridge, MA, 1995.
- [21] Hans-Joachim Hein, Claude LeBrun, *Mass in Kähler Geometry*, arXiv:1507.08885.
- [22] B Kleiner, J Lott, *Notes on Perelman's papers*, Geometry & Topology 12 (2008) 2587-2855.
- [23] B Kleiner, J Lott, *Singular Ricci flows I*, arXiv:1408.2271.
- [24] J. M. Lee, T. H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. 17, no.1, 37-91 (1987).
- [25] P.Li, S.T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156 (1986), 153-201.
- [26] Donovan McFeron and Gábor Székelyhidi, *On the positive mass theorem for manifolds with corners*, Comm.Math.Phys.313(2012), no. 2, 425-443. MR2942956.
- [27] W. Morgan, G. Tian, *Ricci Flow and the Poincare Conjecture*, American Mathematical Society, (2007).
- [28] Mikhail Feldman, Tom Ilmanen, Dan Knopf, *Rotationally symmetric shrinking and expanding gradient Kahler-Ricci solitons*, J. Differential Geom., 65(2):169-209, 2003.
- [29] O. Munteanu, J. Wang, *Smooth metric measure spaces with nonnegative curvature*, Comm. Anal. Geom. 19 (2011), no.3, 451-486.
- [30] Ni, L, *A note on Perelman's LYH inequality*, arXiv:math.DG/0602337.
- [31] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159.
- [32] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109.
- [33] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math/0307245v1.
- [34] O. S. Rothe, *Logarithmic Sobolev inequalities and the spectrum of Schrodinger operators*, J. Func. Anal. 42 (1981), 110-120.
- [35] Robert Haslhofer, *A mass-decreasing flow in dimension three*, Math. Res. Lett. 19(4):927-938, 2012.

- [36] W.X.Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J.Diff. Geom., 30(1989)303-394.
- [37] R.Schoen, S.T.Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. 65, no. 1, 45-76 (1979).
- [38] R.Schoen, S.T.Yau, *Lectures on differential geometry*, Copyright 1994, 2010 by International Press, Somerville, Massachusetts, U.S.A.
- [39] Natasa Sesum, Gang Tian, Xiaodong Wang, *Notes on Perelman's paper on the entropy formula for the Ricci flow and its geometric applications*, <http://www.math.msu.edu/~xwang/perel.pdf>.
- [40] Gang Tian, Jeff Viaclovsky, *Bach-flat asymptotically locally Euclidean metrics*, Inventiones Mathematicae 160 (2), 2005, pages 357-415.
- [41] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. 80, no. 3, 381-402 (1981).
- [42] Qi S. Zhang, *Strong non-collapsing and uniform Sobolev inequalities for Ricci flow with surgeries*, arXiv:math.DG/07121329.
- [43] Qi S. Zhang, *Extremal of Log Sobolev inequality and W entropy on noncompact manifolds*, arXiv:1105.1544

E-mail address: yli427@wisc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI, 53706